Global existence and temporal decay in Keller-Segel models coupled to fluid equations

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Abstract

We consider a Keller-Segel model coupled to the incompressible Navier-Stokes equations in spatial dimensions two and three. We establish the local existence of regular solutions and present some blow-up criteria for both cases that equations of oxygen concentration is of parabolic or hyperbolic type. We also prove global existence and decay estimate in time under the some smallness conditions of initial data.
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1 Introduction

In this paper, we consider mathematical models describing the dynamics of oxygen, swimming bacteria, and viscous incompressible fluids in $\mathbb{R}^d$, with $d = 2, 3$. Bacteria or microorganisms often live in fluid, in which the biology of chemotaxis is intimately related to the surrounding physics. Such a model was proposed by Tuval et al.[24] to describe the dynamics of swimming bacteria, *Bacillus subtilis*. We consider the following equations in [24] and set $Q_T = (0, T] \times \mathbb{R}^d$ with $d = 2, 3$:

$$\begin{cases} \partial_t n + u \cdot \nabla n - \Delta n = -\nabla \cdot (\chi(c) n \nabla c), \\ \partial_t c + u \cdot \nabla c - \mu \Delta c = -k(c) n, \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = -n \nabla \phi, \quad \nabla \cdot u = 0 \end{cases} \quad \text{in } (x,t) \in \mathbb{R}^d \times (0, T], \tag{1.1}$$

where $c(t, x) : Q_T \to \mathbb{R}^+$, $n(t, x) : Q_T \to \mathbb{R}^+$, $u(t, x) : Q_T \to \mathbb{R}^d$ and $p(t, x) : Q_T \to \mathbb{R}$ denote the oxygen concentration, cell concentration, fluid velocity, and scalar pressure, respectively. The nonnegative function $k(c)$ denotes the oxygen consumption rate, and the nonnegative function $\chi(c)$ denotes chemotactic sensitivity. Initial data are given by $(n_0(x), c_0(x), u_0(x))$. We study both cases that either $\mu = 1$ or $\mu = 0$ in the equation of oxygen and, for convenience, the case $\mu = 1$ of (1.1) is called parabolic Keller-Segel-Navier-Stokes equations (abbreviated to P-KSNS) and the case $\mu = 0$ is referred as partially parabolic-hyperbolic Keller-Segel-Navier-Stokes equations (abbreviated to PH-KSNS). We will refer the system to the Keller-Segel-Stokes equations (abbreviated to PH-KSS) if the convection term $u \cdot \nabla u$ is absent in $(1.1)_3$.

To describe the fluid motions, we use Boussinesq approximation to denote the effect due to heavy bacteria. The time-independent function $\phi = \phi(x)$ denotes the potential function produced by different physical mechanisms, e.g., the gravitational force or centrifugal force. Thus, $\phi(x) = a x_d$ is one example of gravity force, and $\phi(x) = \phi(|x|) \to 0$ as $|x| \to \infty$ is an example of centrifugal force.

The classical model to describe the motion of cells was suggested by Patlak[19] and Keller-Segel[13, 14]. It consists of a system of the dynamics of cell density $n = n(t,x)$ and the concentration of chemical attractant substance $c = c(t,x)$ and is given as

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\chi\nabla c), \\ \alpha c_t = \Delta c - \tau c + n, \end{cases} \tag{1.2}$$

where χ is the sensitivity and $\tau^{-\frac{1}{2}}$ represents the activation length. The system (1.2) has been extensively studied by many authors and we will not try to give list of results here (see e.g. [11, 12, 17, 18, 25] and references therein).

Our main objective of this paper is to present blow-up criteria of (1.1) in two or three dimensions, unless solutions exist globally in time (see Theorem 1 and Theorem 3 below), and to establish global existence of regular solutions and their decay properties, when certain norm of initial data is sufficiently small (see Theorem 2 and Theorem 4 below).

We mention previously known results related to ours. In [16] local existence of solutions was shown in three dimensional bounded domains and [8] proved the global-in-time existence of the smooth solutions when initial data are close to constant states in $\mathbb{R}^3$ and $\chi(\cdot), k(\cdot)$ satisfy certain conditions. More precisely, [8] showed that if initial data $\|(n_0 - n_\infty,\, c_0,\, u_0)\|_{H^3}$ is sufficiently small, then there exists a unique global solution, provided that

$$\chi'(\cdot) \geq 0, \quad k'(\cdot) > 0, \quad \left(\frac{k(\cdot)}{\chi(\cdot)}\right)'' < 0. \tag{1.3}$$

In the absence of the fluid in (1.1), i.e., $u = 0$, [22] showed that there exists a unique, global and bounded solution if χ is sufficiently small, dependent upon $\|c_0\|_{L^\infty(\mathbb{R}^d)}$.

For two dimensional case, in [15], Liu and Lorz showed the global existence of a weak solution in $\mathbb{R}^2$ under the following conditions on $\chi(\cdot)$ and $k(\cdot)$:

$$\chi'(\cdot) \geq 0, \quad (\chi(\cdot)k(\cdot))' > 0, \quad \left(\frac{k(\cdot)}{\chi(\cdot)}\right)'' < 0. \tag{1.4}$$

In two dimensions, Winkler[26] proved the global existence of regular solutions without smallness assumptions on initial data for bounded domains with boundary conditions $\partial_\nu n = \partial_\nu c = u = 0$ under the following sign conditions on $\chi(\cdot)$ and $k(\cdot)$:

$$\left(\frac{k(\cdot)}{\chi(\cdot)}\right)' > 0, \quad (\chi(\cdot)k(\cdot))' \geq 0, \quad \left(\frac{k(\cdot)}{\chi(\cdot)}\right)'' \leq 0. \tag{1.5}$$

In [1] the authors of the paper established global existence of smooth solutions in $\mathbb{R}^2$ with no smallness of the initial data and a certain conditions, motivated by experimental results in [2] and [24], on $\chi(\cdot)$ and $k(\cdot)$ (compare to (1.5)), that is,

$$\chi(c),\ k(c),\ \chi'(c),\ k'(c) \geq 0, \text{ and } \ \sup|\chi(c) - \mu k(c)| < \epsilon \ \text{ for some } \mu > 0. \tag{1.6}$$

Construction of weak solutions in $\mathbb{R}^3$ was also discussed in [1] with replacement of $|\chi(c) - \mu k(c)| = 0$ in (1.6). We refer to [9], [23] and [3] and references therein for the nonlinear diffusion models of a porous medium type Δn^m, instead of Δn.

As mentioned earlier, our main motivation is to study existence of regular solutions of (1.1) when certain norm of initial data is small. To be more precise, we show that in case $\mu = 1$, if

$\|c_0\|_{L^\infty}$ is small, then solutions become regular in $\mathbb{R}^d$, $d = 2, 3$ and satisfy a certain degree of decay in time (when $d = 3$, Stokes system is under our consideration for fluid equations). On the other hand, in case $\mu = 0$, we establish local solutions in time, and then global solutions with time decay if $\|n_0\|_{L^{\frac{d}{2}}}$ is small. We first consider the case (P-KSF) in (1.1). Local-in-time existence of classical solutions for (1.1) was established in Theorem 1 in [1] such that $(n, c, u) \in L^\infty(0, T; H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d))$ for some $T > 0$ and $m \geq 3$. We present some blow-up criteria of local classical solutions, unless the maximal existence time is infinite.

Theorem 1 *Let the initial data (n_0, c_0, u_0) be given in $H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d)$ for $m \geq 3$ and $d = 2, 3$. Assume that χ, k, χ', k' are all non-negative and $\chi,\ k \in C^m(\mathbb{R}^+)$ and $k(0) = 0$, $\|\nabla^l \phi\|_{L^\infty} < \infty$ for $1 \leq |l| \leq m$. If the maximal time of existence, T^*, in Theorem 1 in [1], is finite, then one of the following is true in each case of $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively:*

$$(2D) \qquad \|n\|_{L^q(0,T^*;L^p(\mathbb{R}^2))} = \infty, \qquad \frac{2}{p} + \frac{2}{q} = 2,\ 1 < l \leq \infty. \tag{1.7}$$

$$(3D) \qquad \|u\|_{L^\gamma(0,T^*;L^\beta(\mathbb{R}^3))} + \|n\|_{L^q(0,T^*;L^p(\mathbb{R}^3))} = \infty, \tag{1.8}$$

where

$$\frac{3}{\beta} + \frac{2}{\gamma} \leq 1,\ 3 < \beta \leq \infty, \quad \frac{3}{p} + \frac{2}{q} = 2,\ \frac{3}{2} < p \leq \infty.$$

If fluid equation is the Stokes system for $(3D)$, $\|u\|_{L^q(0,T^*;L^p(\mathbb{R}^3))}$ in (1.8) is dropped.

The proof of Theorem 1 will be given in section 2.

Remark 1 *We remind the following scaling invariance of (1.1):*

$$n_R(t, x) := R^2 n(R^2 t, Rx), \quad c_R(t, x) = c(R^2 t, Rx), \quad u_R(t, x) = Ru(R^2 t, Rx) \tag{1.9}$$

and observe that (1.7) and (1.8) are invariant functionals under the scaling (1.9). For the limiting case $(l, m) = (1, \infty)$ in (1.7), due to conservation of total mass, we note that $\|n\|_{L^\infty(0,t;L^1(\mathbb{R}^2))} = \|n_0\|_{L^1(\mathbb{R}^2)} < \infty$ for any $t < T^*$. In Proposition 1 in section 2, we prove that if $\|n_0\|_{L^1(\mathbb{R}^2)}$ is sufficiently small, blow-up does not occur in a finite time. We, however, leave an open question whether or not singularity may develop for large L^1 norm of n_0.

Remark 2 *Liu and Lorz[15] showed global-in-time existence of weak solution to (1.1) in two dimensional case under the assumption (1.4). Since their weak solution satisfies integrability $n \in L^2(0, T; L^2(\mathbb{R}^2))$ for any $T > 0$, which is a special case in (1.7), their weak solution is, in fact, a classical solution if $(n_0, c_0, u_0) \in H^{m-1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d) \times H^m(\mathbb{R}^d)$ for $m \geq 3$ as a consequence of Theorem 1.*

The second result is the existence of regular solutions under the assumption that $\|c_0\|_{L^\infty}$ is sufficiently small.

Theorem 2 *Let the assumptions in Theorem 1 hold. We consider the Navier-Stokes equations in $\mathbb{R}^2$ and the Stokes system in $\mathbb{R}^3$ in $(1.1)_3$. There exists a constant $\delta > 0$ such that if $\|c_0\|_{L^\infty(\mathbb{R}^d)} < \delta$, then classical solution of (1.1) exists globally. Furthermore, n and c satisfy the following time decay:*

$$\|n(t)\|_{L^\infty(\mathbb{R}^d)} + \|c(t)\|_{L^\infty(\mathbb{R}^d)} \leq C(1 + t)^{-\frac{d}{4}}, \qquad d = 2, 3. \tag{1.10}$$

The proof of Theorem 2 will be given in section 3. Next we study (PP-KSF), namely

$$\begin{cases} \partial_t n + u \cdot \nabla n - \Delta n = -\nabla \cdot [\chi(c) n \nabla c] \\ \partial_t c + u \cdot \nabla n = -k(c) n \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = -n \nabla \phi, \quad \nabla \cdot u = 0. \end{cases} \qquad t > 0, \ x \in \mathbb{R}^d \tag{1.11}$$

When the fluid is absent, the Keller-Segel equations with chemical of ODE type, typically referred to the angiogenesis system, has been studied in [4, 5, 6] and [20]:

$$\partial_t n - \Delta n = -\nabla \cdot [\chi(c) n \nabla c], \quad \partial_t c = -c^m n \qquad t > 0, \ x \in \mathbb{R}^d. \tag{1.12}$$

In section 4, we show local classical solution of (1.11) by the usual iteration method and present blow-up criteria of (1.11), if a finite time singularity occurs. Now we state the third main result.

Theorem 3 *Let $m \geq 3$ and $d = 2, 3$. Assume that $\chi(\cdot), k(\cdot) \in C^m(\mathbb{R})$, $\left\|\nabla^l \phi\right\|_{L^\infty} < \infty$ for $1 \leq |l| \leq m$. Then there exists T^*, the maximal existence time, such that if $(n_0, c_0, u_0) \in H^m(\mathbb{R}^d) \times H^{m+1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d)$, then there exists a unique classical solution of* (1.11) *satisfying for any $t < T^*$*

$$(n, c, u) \in C(0, t; H^m(\mathbb{R}^d) \times H^{m+1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d)),$$
$$(n, u) \in L^2(0, t; H^{m+1}(\mathbb{R}^d) \times H^{m+1}).$$

Furthermore, if $T^* < \infty$, then one of the following is true in each case of $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively:

$$(2D) \qquad \|n\|_{L^2(0,T^*;L^\infty(\mathbb{R}^2))} = \infty \tag{1.13}$$

$$(3D) \qquad \|u\|_{L^\gamma(0,T^*;L^\beta(\mathbb{R}^3))} + \|n\|_{L^2(0,T^*;L^\infty(\mathbb{R}^3))} = \infty, \qquad \frac{3}{\beta} + \frac{2}{\gamma} \leq 1, \ 3 < \beta \leq \infty. \tag{1.14}$$

If fluid equation is the Stokes system for $(3D)$, $\|u\|_{L^q(0,T^*;L^p(\mathbb{R}^3))}$ in (1.14) *is dropped.*

Last main result is global existence of regular solutions for (1.11) and their decays in time, when $\|n_0\|_{L^{\frac{d}{2}}(\mathbb{R}^d)}$ is sufficiently small. To be more precise, we obtain the following:

Theorem 4 *Let $d = 2, 3$ and we consider the Navier-Stokes equations in $\mathbb{R}^2$ and the Stokes system in $\mathbb{R}^3$ in $(1.11)_3$. Suppose that $(n_0, c_0, u_0) \in H^m(\mathbb{R}^d) \times H^{m+1}(\mathbb{R}^d) \times H^m(\mathbb{R}^d))$ for $m \geq 3$. Then there exists $\epsilon_2 = \epsilon_2(d, \|c_0\|_{L^\infty})$ such that if $\|n_0\|_{L^{\frac{d}{2}}(R^d)} < \epsilon_2$, then solutions of* (1.11) *become global and classical. Furthermore, n satisfies the following time decay:*

$$\|n(t)\|_{L^\infty} \leq C(1 + t)^{-1}. \tag{1.15}$$

This paper is organized as follows. In Section 2, we prove Theorem 1 by obtaining a priori estimates. Section 3 is devoted to prove Theorem 2 by adjusting De Giorgi method introduced in [20]. We obtain the blow-up criteria in Theorem 3 by using a priori energy estimates in Section 4 and The proof of Theorem 4 is presented again by using De Giorgi method in the last section.

2 Blow-up criteria of parabolic system

We first recall following blow-up criteria for (1.1) obtained in [1, Theorem 2]:

$$(2D) \qquad \int_0^{T^*} \|\nabla c\|^2_{L^\infty(\mathbb{R}^2)} = \infty, \tag{2.1}$$

$$(3D) \qquad \int_0^{T^*} \|u\|^\gamma_{L^\beta(\mathbb{R}^3)} + \int_0^{T^*} \|\nabla c\|^2_{L^\infty(\mathbb{R}^3)} = \infty, \qquad \frac{3}{\beta}+\frac{2}{\gamma}=1,\ 3<\beta\le\infty, \tag{2.2}$$

where T^* is the maximal time of existence. If the fluid equation is the Stokes system, not the Navier-Stokes equations, in case of $3D$, then the condition on v in (2.2) is not necessary and thus it can be dropped. From now on, we denote $L^{q,p}_{t,x} = L^q(0,T^*;L^p(\mathbb{R}^d))$ and $L^p_{t,x} = L^p(0,T^*;L^p(\mathbb{R}^d))$, unless any confusion is to be expected. All generic constants will be written by C, which may change from one line to the other and ϵ will be used to indicate some sufficiently small positive number.

Proof of Theorem 1 We argue by contradiction. We suppose that (1.7) in 2D and (1.8) in 3D are finite. We then show that T^* cannot be a finite maximal time of existence, which will lead to a contradiction. We start with the case of dimension two.

• (**2D case**) We first suppose $\|n\|_{L^2_{t,x}([0,T^*)\times\mathbb{R}^2)}$ is finite. We will show $\int_0^{T^*}\|\nabla c\|^2_{L^\infty}dt<\infty$, which is contrary to the blow-up criterion (2.1) proved in [1]. In the following, we obtain a priori estimates, since our computations are made for any time T with $T<T^*$. We frequently use the following type of interpolation inequality

$$\|D^l f\|_{L^p(\mathbb{R}^2)} \le C\|f\|^\theta_{L^q(\mathbb{R}^2)}\|D^k f\|^{1-\theta}_{L^r(\mathbb{R}^2)}, \tag{2.3}$$

where $0\le l\le k-1$ and $l-\frac{2}{p} = -\theta\frac{2}{q}+(1-\theta)(k-\frac{2}{r})$ with $0\le\theta\le 1$. From maximum principle for c and conservation of mass for n, it is immediate that $\|c\|_{L^\infty_{t,x}}\le\|c_0\|_{L^\infty}$ and $\|n\|_{L^{\infty,1}_{t,x}}\le\|n_0\|_{L^1}$. We note that the convection term, $(u\cdot\nabla)c$, is estimated as follows:

$$\begin{aligned}\|(u\cdot\nabla)c\|^2_{L^2_{t,x}} &\le C\int_0^T \|u\|_{L^2}\|\nabla u\|_{L^2}\|c\|_{L^\infty}\|\Delta c\|_{L^2}dt\\ &\le \epsilon\|\Delta c\|^2_{L^2_{t,x}} + C_\epsilon\|u\|^2_{L^{\infty,2}_{t,x}}\|c\|^2_{L^\infty_{t,x}}\|\nabla u\|^2_{L^2_{t,x}}.\end{aligned} \tag{2.4}$$

Via L^2-estimate of the heat equation, we have

$$\|c_t\|^2_{L^2_{t,x}} + \|\Delta c\|^2_{L^2_{t,x}} \le C\|c_0\|^2_{H^1} + C\|n\|^2_{L^2_{t,x}} + \|(u\cdot\nabla)c\|^2_{L^2_{t,x}}. \tag{2.5}$$

Combining (2.4) and (2.5) together with the hypothesis $\|n\|^2_{L^2_{t,x}}<\infty$, we obtain $\|\Delta c\|_{L^2_{t,x}}<\infty$. On the other hands, L^2 scalar product for equation of n gives that

$$\frac{d}{dt}\|n\|^2_{L^2} + \|\nabla n\|^2_{L^2} \le C\|n\nabla c\|_{L^2}\|\nabla n\|_{L^2} \le \frac14\|\nabla n\|^2_{L^2} + C\|n\|^2_{L^4}\|\nabla c\|^2_{L^4}$$

$$\le \frac14\|\nabla n\|^2_{L^2} + C\|n\|_{L^2}\|\nabla n\|_{L^2}\|c\|_{L^\infty}\|\Delta c\|_{L^2} \le \frac12\|\nabla n\|^2_{L^2} + C\|c\|^2_{L^\infty}\|\Delta c\|^2_{L^2}\|n\|^2_{L^2}.$$

Using Gronwall's inequality, we have

$$\|n\|^2_{L^{\infty,2}_{t,x}} + \|\nabla n\|^2_{L^2_{t,x}} \le \|n_0\|^2_{L^2} \exp\left(C\|c\|^2_{L^\infty_{t,x}} \|\Delta c\|^2_{L^2_{t,x}} \right).$$

Next, testing $-\Delta c$ with equation of c, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla c\|^2_{L^2} + \|\Delta c\|^2_{L^2} \le \int_{\mathbb{R}^2} |\nabla u||\nabla c|^2 dx + C\|n\|^2_{L^2} + \frac{1}{4}\|\Delta c\|^2_{L^2}$$

$$\le C\|\nabla u\|_{L^2}\|\nabla c\|^2_{L^4} + C\|n\|^2_{L^2} + \frac{1}{4}\|\Delta c\|^2_{L^2} \le C\|\nabla u\|^2_{L^2}\|\nabla c\|^2_{L^2} + C\|n\|^2_{L^2} + \frac{1}{2}\|\Delta c\|^2_{L^2}.$$

Again, Gronwall's inequality impiles that

$$\|\nabla c\|^2_{L^{\infty,2}_{t,x}} + \|\Delta c\|^2_{L^2_{t,x}} < \infty.$$

The energy estimate of vorticity equation leads to

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2_{L^2} + \|\nabla \omega\|^2_{L^2} \le C\|n\|^2_{L^2} + \frac{1}{4}\|\nabla \omega\|^2_{L^2}.$$

Therefore, we obtain $\|\omega\|^2_{L^{\infty,2}_{t,x}} + \|\nabla \omega\|^2_{L^{2,2}_{t,x}} < \infty$ via Gronwall's inequality. Finally, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta c\|^2_{L^2} + \|\nabla \Delta c\|^2_{L^2} \le C\|\nabla u \nabla c\|^2_{L^2} + C\|u\nabla^2 c\|^2_{L^2} + C\|\nabla c n\|^2_{L^2} + C\|\nabla n\|^2_{L^2}$$

$$\le C\|\nabla u\|^2_{L^3}\|\nabla c\|^2_{L^6} + C\|u\|^2_{L^\infty}\|\Delta c\|^2_{L^2} + C\|\nabla c\|^2_{L^6}\|n\|^2_{L^3} + C\|\nabla n\|^2_{L^2}.$$

This gives the bound of $\|\nabla \Delta c\|^2_{L^{2,2}_{t,x}}$ and in turn, the bound of $\|\nabla c\|_{L^{2,\infty}_{t,x}}$. We complete the proof for the case $\|n\|_{L^2_{t,x}} < \infty$.

In case that $\|n\|_{L^{p,q}_{x,t}} < \infty$ with $\frac{2}{p} + \frac{2}{q} \le 2$ and $p > 2$, we observe that $\|n\|_{L^2_{x,t}} < \infty$. Indeed, due to interpolation and Hölder's inequality,

$$\int_0^T \|n\|^2_{L^2} dt \le C \int_0^T \|n\|^{\frac{p-2}{p-1}}_{L^1} \|n\|^{\frac{p}{p-1}}_{L^p} dt \le C\|n\|^{\frac{p}{p-1}}_{L^{p,q}_{x,t}}.$$

Hence, with the aid of previous result of $L^2_{x,t}$ case, the case $\frac{2}{p} + \frac{2}{q} \le 2$ with $p > 2$ is direct. It remains to consider the case $\|n\|_{L^{p,q}_{x,t}} < \infty$ with $\frac{2}{p} + \frac{2}{q} = 2$ and $1 < p < 2$. We note first that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2} \le C\|n\|_{L^p}\|u\|_{L^{\frac{p}{p-1}}}$$

$$\le C\|n\|_{L^p}\|u\|^{\frac{2(p-1)}{p}}_{L^2}\|\nabla u\|^{\frac{2-p}{p}}_{L^2} \le C\|n\|^{\frac{2p}{3p-2}}_{L^p}\|u\|^{\frac{4(p-1)}{3p-2}}_{L^2} + \frac{1}{2}\|\nabla u\|^2_{L^2}.$$

Therefore, due to Gronwall's inequality, $\sup \|u\|^2_{L^2} + \int_0^T \|\nabla u\|^2_{L^2} dt < \infty$.
Using the mixed norm estimate of the heat equation for c, we have

$$\|c_t\|^q_{L^{p,q}} + \|\Delta c\|^q_{L^{p,q}} \le C\|c_0\|^q_{H^2} + C\|u \cdot \nabla c\|^q_{L^{p,q}} + C\|n\|^q_{L^{p,q}}. \tag{2.6}$$

Noting that $\frac{(p-1)q}{p} = 1$, we compute

$$\int_0^T \|u\cdot\nabla c\|_{L^p}^q dt \le \int_0^T \|u\|_{L^{2p}}^q \|\nabla c\|_{L^{2p}}^q dt \le C\int_0^T \|u\|_{L^{2p}}^q \|c\|_{L^\infty}^{\frac{q}{2}} \|\Delta c\|_{L^p}^{\frac{q}{2}} dt$$

$$\le C\int_0^T \|u\|_{L^{2p}}^{2q} dt + \epsilon\int_0^T \|\Delta c\|_{L^p}^q dt \le C\|u\|_{L^{2,\infty}_{x,t}}^{\frac{2q}{p}} \int_0^T \|\nabla u\|_{L^2}^{\frac{2(p-1)q}{p}} dt + \epsilon\int_0^T \|\Delta c\|_{L^p}^q dt. \quad (2.7)$$

Combining (2.6) and (2.7) with sufficiently small $\epsilon > 0$, we have

$$\|c_t\|_{L^{p,q}}^q + \|\Delta c\|_{L^{p,q}}^q \le C.$$

Multiplying the equation of n with $\ln n$ and integrating it by parts, we obtain

$$\frac{d}{dt}\int n\ln n dx + \|\nabla\sqrt{n}\|_{L^2}^2 \le C\|\nabla c\|_{L^{\frac{2p}{2-p}}} \|\nabla\sqrt{n}\|_{L^2} \|\sqrt{n}\|_{L^{\frac{p}{p-1}}}$$

$$\le C\|\Delta c\|_{L^p} \|\sqrt{n}\|_{L^2}^{\frac{2(p-1)}{p}} \|\nabla\sqrt{n}\|_{L^2}^{\frac{2}{p}} \le C\|\Delta c\|_{L^p}^q \|\sqrt{n}\|_{L^2}^2 + \frac{1}{2}\|\nabla\sqrt{n}\|_{L^2}^2. \quad (2.8)$$

Using Gronwall inequality, the estimate (2.8) leads to $\nabla\sqrt{n} \in L^2_{t,x}$, which implies $\|n\|_{L^2_{t,x}} < \infty$. This completes the proof for 2D case.

• (**3D case**) Suppose that (1.8) is not true. As in 2D case, we then show $\int_0^{T^*} \|\nabla c\|_{L^\infty(\mathbb{R}^3)} < \infty$, which is contrary to the blow-up criterion (2.2) proved in [1]. The proof of the case for Stokes system is omitted, since its verification is simpler. We first show $\nabla c \in L^\infty(0, T^*; L^2)$ and $\nabla^2 c \in L^2(0, T^*; L^2)$. Testing $\log n$ to the equation $(1.1)_1$,

$$\frac{d}{dt}\int n\log n + 4\int \left|\nabla n^{\frac{1}{2}}\right|^2 = \int \chi(c)\nabla n\nabla c \le C\int \left|\nabla n^{\frac{1}{2}}\right| n^{\frac{1}{2}} |\nabla c|$$

$$\le C\left\|\nabla n^{\frac{1}{2}}\right\|_{L^2} \left\|n^{\frac{1}{2}}\right\|_{L^{2p}} \|\nabla c\|_{L^{\frac{2p}{p-1}}} \le \epsilon\left\|\nabla n^{\frac{1}{2}}\right\|_{L^2}^2 + C_\epsilon \|n\|_{L^p} \|\nabla c\|_{L^{\frac{2p}{p-1}}}^2. \quad (2.9)$$

Via Gagliardo-Nirenberg's inequality, we note $\|\nabla c\|_{L^{\frac{2p}{p-1}}} \le C\|\nabla c\|_{L^2}^{\frac{2p-3}{2p}} \left\|\nabla^2 c\right\|_{L^2}^{\frac{3}{2p}}$, and therefore, combining (2.9), we obtain

$$\frac{d}{dt}\int n\log n + \int \left|\nabla n^{\frac{1}{2}}\right|^2 \le \epsilon\left\|\nabla^2 c\right\|_{L^2}^2 + C_\epsilon \|n\|_{L^p}^{\frac{2p}{2p-3}} \|\nabla c\|_{L^2}^2. \quad (2.10)$$

Multiplying $(1.1)_1$ with $-\Delta c$ and using integration by parts,

$$\frac{1}{2}\frac{d}{dt}\int |\nabla c|^2 + \int |\Delta c|^2 = \int u\nabla c\Delta c + \int k(c)n\Delta c := I + J. \quad (2.11)$$

We first consider the term J. Following the same computations in (2.9)-(2.10),

$$J = -\int k'(c)n|\nabla c|^2 - \int k\nabla n\nabla c \le \epsilon\|n^{\frac{1}{2}}\|_{L^2}^2 + \epsilon\left\|\nabla^2 c\right\|_{L^2}^2 + C_\epsilon \|n\|_{L^p}^{\frac{2p}{2p-3}} \|\nabla c\|_{L^2}^2, \quad (2.12)$$

where we used that $k' \ge 0$. On the other hand, I is estimated as follows:

$$I \le \|u\|_{L^\beta} \|\nabla c\|_{L^{\frac{2\beta}{\beta-2}}} \|\Delta c\|_{L^2} \le C\|u\|_{L^\beta} \|\nabla c\|_{L^2}^{\frac{\beta-3}{\beta}} \left\|\nabla^2 c\right\|_{L^2}^{\frac{\beta+3}{\beta}}$$

$$\leq \epsilon \left\|\nabla^2 c\right\|_{L^2}^2 + C_\epsilon \left\|u\right\|_{L^\beta}^{\frac{2\beta}{\beta-3}} \left\|\nabla c\right\|_{L^2}^2 \tag{2.13}$$

Summing up (2.10), (2.12) and (2.13),

$$\frac{d}{dt}(\int n\log n + \frac{1}{2}\left\|\nabla c\right\|_{L^2}^2) + \left\|\nabla n^{\frac{1}{2}}\right\|_{L^2}^2 + \left\|\nabla^2 c\right\|_{L^2}^2 \leq C_\epsilon(\left\|n\right\|_{L^p}^{\frac{2p}{2p-3}} + \left\|u\right\|_{L^\beta}^{\frac{2\beta}{\beta-3}}) \left\|\nabla c\right\|_{L^2}^2. \tag{2.14}$$

Due to Gronwall inequality, we observe that $\nabla c \in L^\infty(0,T^*;L^2)$ and $\nabla^2 c \in L^2(0,T^*;L^2)$.

Next we consider the vorticity equation of fluid equations in (1.1)

$$\partial_t \omega - \Delta\omega + u\cdot\nabla\omega - \omega\cdot\nabla u = -\nabla\times(n\nabla\phi).$$

Energy estimate shows

$$\frac{1}{2}\frac{d}{dt}\left\|\omega\right\|_{L^2}^2 + \left\|\nabla\omega\right\|_{L^2}^2 = \int \omega\cdot\nabla u\omega - \int \nabla\times(n\nabla\phi)\omega := K_1 + K_2. \tag{2.15}$$

First, we estimate K_2. Following computations as in above,

$$K_2 \leq C\int |\nabla n|\,|\omega| \leq C\left\|\nabla n^{\frac{1}{2}}\right\|_{L^2}\left\|n^{\frac{1}{2}}\right\|_{L^{2p}}\left\|\omega\right\|_{L^{\frac{2p}{p-1}}} \leq C\left\|\nabla n^{\frac{1}{2}}\right\|_{L^2}\left\|n\right\|_{L^p}^{\frac{1}{2}}\left\|\omega\right\|_{L^2}^{\frac{2p-3}{2p}}\left\|\nabla\omega\right\|_{L^2}^{\frac{3}{2p}}$$

$$\leq \epsilon\left\|\nabla n^{\frac{1}{2}}\right\|_{L^2}^2 + \epsilon\left\|\nabla\omega\right\|_{L^2}^2 + C_\epsilon\left\|n\right\|_{L^p}^{\frac{2p}{2p-3}}\left\|\omega\right\|_{L^2}^2, \tag{2.16}$$

On the other hand, for K_1, similarly as in (2.13), we show

$$K_1 \leq \epsilon\left\|\nabla\omega\right\|_{L^2}^2 + C_\epsilon\left\|u\right\|_{L^\beta}^{\frac{2\beta}{\beta-3}}\left\|\omega\right\|_{L^2}^2. \tag{2.17}$$

Adding above estimate together and using Gronwall inequality, we observe that $\omega \in L^\infty(0,T^*;L^2)$ and $\nabla\omega \in L^2(0,T^*;L^2)$.

Next, considering four cases of $\frac{3}{2} < p < 2$, $2 \leq p \leq 3$, $3 < p \leq 6$, and $p > 6$ separately, we will show that

$$\|u\nabla c\|_{L^q(0,T^*;L^p)} \leq C + \epsilon\|\nabla^2 c\|_{L^q(0,T^*;L^p)}. \tag{2.18}$$

The proof of (2.18) will be given later. Then by the maximal regularity of heat equation,

$$\|c_t\|_{L^q(0,T^*;L^p)} + \|\Delta c\|_{L^q(0,T^*;L^p)} \leq C + \epsilon\|\nabla^2 c\|_{L^q(0,T^*;L^p)} + C\|n\|_{L^q(0,T^*;L^p)}, \tag{2.19}$$

we obtain $\|\nabla^2 c\|_{L^q(0,T^*;L^p)} < \infty$. Now we turn to show that $n \in L^\infty(0,T^*;L^r)$ for any $r > 1$. Testing n^{r-1} to $(1.1)_1$ and noting $\chi' \geq 0$, we observe that

$$\frac{1}{r}\frac{d}{dt}\left\|n\right\|_{L^r}^r + \frac{4(r-1)}{r^2}\left\|\nabla n^{\frac{r}{2}}\right\|_{L^2}^2 = -\frac{r-1}{r}\left(\int \chi'\left|\nabla c\right|^2 n^r + \int \chi\Delta c n^r\right) \leq C\int\left|\Delta c\right| n^r$$

$$\leq C\left\|\Delta c\right\|_{L^p}\left\|n^r\right\|_{L^{\frac{p}{p-1}}} \leq C\left\|\Delta c\right\|_{L^p}\left\|n^r\right\|_{L^1}^{\frac{2p-3}{2p}}\left\|n^r\right\|_{L^3}^{\frac{3}{2p}}$$

$$\leq C\left\|\Delta c\right\|_{L^p}\left\|n^r\right\|_{L^1}^{\frac{2p-3}{2p}}\left\|\nabla n^{\frac{r}{2}}\right\|_{L^2}^{\frac{3}{l}} \leq C_\epsilon\left\|\Delta c\right\|_{L^p}^{\frac{2p}{2p-3}}\left\|n\right\|_{L^r}^r + \epsilon\left\|\nabla n^{\frac{r}{2}}\right\|_{L^2}^2. \tag{2.20}$$

Since $\nabla^2 c \in L^q(0,T^*;L^p)$, via Gronwall inequality, we can prove that $n \in L^\infty(0,T^*;L^r)$ for any $r > 1$. Let us choose $r > 3$ and via (2.19) we then obtain that $\|\nabla c\|_{L^2_t L^\infty_x} < \infty$. Indeed, we

note that $\|u\nabla c\|_{L^r} < C_\epsilon + \epsilon\|\nabla^2 c\|_{L^r}$ for $3 < r \le 6$ (see (2.23) below). Again by the maximal regularity of heat equation, we have

$$\|c_t\|_{L^2(0,T^*;L^r)} + \|\Delta c\|_{L^2(0,T^*;L^r)} \le C_\epsilon + \epsilon\|\nabla^2 c\|_{L^2(0,T^*;L^r)}.$$

Combined with $\|\nabla c\|_{L^\infty_t L^2_x} + \|\nabla^2 c\|_{L^2_{t,x}} < \infty$, the above yields to $\|\nabla c\|_{L^2_t L^\infty_x} < \infty$ as desired, which is contrary to a blow-up criterion (2.2).

It remains to show the estimate (2.18). By $(\nabla c, \omega) \in L^\infty_t L^2_x$ and $(\nabla^2 c, \nabla\omega) \in L^2_{t,x}$, it follows that $u\nabla c$ belongs to $L^4(0,T^*;L^2) \cap L^2(0,T^*;L^3)$ and so $u\nabla c \in L^q(0,T^*;L^p)$ for $2 \le p \le 3$. The other cases are treated as follows.

(i) (Case $\frac{3}{2} < p < 2$) Setting $p^* = 3p/(3-p)$, we have

$$\|u\nabla c\|_{L^p} \le \|u\|_{L^6} \|\nabla c\|_{L^{\frac{6p}{6-p}}} \le C \|\omega\|_{L^2} \|\nabla c\|_{L^2}^{\frac{p}{5p-6}} \|\nabla c\|_{L^{p^*}}^{\frac{4p-6}{5p-6}} \le C_\epsilon + \epsilon\|\nabla^2 c\|_{L^p},$$

where we use that $2 < 6p/(6-p) < p^*$. Taking L^q-norm in time variable, it follows that $u\nabla c \in L^q(0,T^*;L^p)$.

(ii) (Case $3 < p \le 6$) Using that ω and ∇c are in $L^\infty(0,T^*;L^2)$, we have

$$\begin{aligned}\|u\nabla c\|_{L^p} &\le \|u\|_{L^2}^{\frac{6-p}{2p}} \|u\|_{L^6}^{\frac{3(p-2)}{2p}} \|\nabla c\|_{L^\infty} \le C \|u\|_{L^2}^{\frac{6-p}{2p}} \|\omega\|_{L^2}^{\frac{3(p-2)}{2p}} \|\nabla c\|_{L^2}^{\frac{2(p-3)}{5p-6}} \left\|\nabla^2 c\right\|_{L^p}^{\frac{3p}{5p-6}}\\ &\le C \|u\|_{L^2}^{\frac{6-p}{2p}} \left\|\nabla^2 c\right\|_{L^p}^{\frac{3p}{5p-6}} \le C_\epsilon \|u\|_{L^2}^{\frac{5p-6}{2(p-3)}} + \epsilon \left\|\nabla^2 c\right\|_{L^p}.\end{aligned}$$

On the other hands, due to $p/(p-1) \le q = 2p/(2p-3)$, we note that

$$\|n\|_{L^2}^2 \le \|n\|_{L^1}^{\frac{p-2}{p-1}} \|n\|_{L^p}^{\frac{p}{p-1}} \le C + C \|n\|_{L^p}^q. \tag{2.21}$$

From $(1.1)_3$, we have

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \le C\|u\|_{L^2}^2 + \|n\|_{L^2}^2. \tag{2.22}$$

Combining (2.21) and (2.22), we obtain $\|u\|_{L^\infty_t L^2_x} + \|\nabla u\|_{L^2_{t,x}} < \infty$. Therefore we obtain

$$\|u\nabla c\|_{L^p} \le C_\epsilon + \epsilon \left\|\nabla^2 c\right\|_{L^p} \qquad \text{for any } \ t < T^*. \tag{2.23}$$

(iii) (Case $p > 6$) We estimate

$$\begin{aligned}\|u\nabla c\|_{L^p} &\le \|u\|_{L^p} \|\nabla c\|_{L^\infty} \le C \|u\|_{L^\infty}^{\frac{p-2}{p}} \|u\|_{L^2}^{\frac{2}{p}} \|\nabla c\|_{L^2}^{\frac{2(p-3)}{5p-6}} \left\|\nabla^2 c\right\|_{L^p}^{\frac{3p}{5p-6}} \le C \|u\|_{L^\infty}^{\frac{p-2}{p}} \left\|\nabla^2 c\right\|_{L^p}^{\frac{3p}{5p-6}}\\ &\le C \|\nabla u\|_{L^2}^{\frac{p-2}{2p}} \left\|\nabla^2 u\right\|^{\frac{p-2}{2p}} \left\|\nabla^2 c\right\|_{L^p}^{\frac{3p}{5p-6}} \le C \|\nabla\omega\|_{L^2}^{\frac{p-2}{2p}} \left\|\nabla^2 c\right\|_{L^p}^{\frac{3p}{5p-6}}.\end{aligned}$$

Therefore, suing $q = 2p/(2p-3)$ and Young's inequality, we have

$$\|u\nabla c\|_{L^p}^q \le C_\epsilon \|\nabla\omega\|_{L^2}^2 + \epsilon \left\|\nabla^2 c\right\|_{L^p}^{\frac{12p^2}{(5p-6)(3p-4)}} \le C_\epsilon(1 + \|\nabla\omega\|_{L^2}^2) + \epsilon \left\|\nabla^2 c\right\|_{L^p}^q, \tag{2.24}$$

where we used that $\frac{12p^2}{(5p-6)(3p-4)} + < q = \frac{2p}{2p-3}$ with $p > 6$. Therefore, the estimate (2.18) is also true for $p > 6$. This completes the proof. $\square$

Next we present the proof of existence of regular solutions in case $\|n_0\|_{L^1}$ is small in dimension two.

Proposition 1 *Let $d = 2$ and initial data, χ, k, and ϕ satisfy the assumptions in Theorem 1. Assume further that $\|n_0 \ln n_0\|_{L^1(\mathbb{R}^2)} + \|\langle x\rangle n_0\|_{L^1(\mathbb{R}^2)}$ is finite. Then, there exists an $\epsilon > 0$ such that if $\|n_0\|_{L^1} < \epsilon$ then the maximal time of existence, T^*, is infinite, i.e. $T^* = \infty$.*

Proof. Estimates in this proof are a priori, since all computations are made before the maximal time of existence, T^*. We note that, due to the conservation of mass, $\sup_{0\le t<T^*} \|n\|_{L^1} < \epsilon$ and therefore, we have

$$\|n\|_{L^2} \le C\|\sqrt{n}\|_{L^2}\|\nabla\sqrt{n}\|_{L^2} \le C\epsilon\|\nabla\sqrt{n}\|_{L^2}. \tag{2.25}$$

Multiplying equations of n, c in (1.1) with $\ln n$, Δc, respectively, we obtain

$$\frac{d}{dt}\int n\ln n dx + \int |\nabla\sqrt{n}|^2 dx = -\int \chi(c)\Delta c n dx \le C\epsilon\|\Delta c\|_{L^2}\|\nabla\sqrt{n}\|_{L^2}. \tag{2.26}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla c\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 &\le C\epsilon\|\nabla\sqrt{n}\|_{L^2}\|\Delta c\|_{L^2} + C\|\nabla u\|_{L^4}\|\nabla c\|_{L^4}\|\nabla c\|_{L^2}\\ &\le C\epsilon\|\nabla\sqrt{n}\|_{L^2}^2 + \frac{1}{2}\|\Delta c\|_{L^2}^2 + C\|\nabla u\|_{L^2}\|\nabla c\|_{L^2}^3 + \frac{1}{4}\|\nabla\omega\|_{L^2}^2.\end{aligned} \tag{2.27}$$

Adding (2.26) and (2.27) with the following estimate:

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 \le C\|n\|_{L^2}^2 + \frac{1}{4}\|\nabla\omega\|_{L^2}^2,$$

we have

$$\begin{aligned}\frac{d}{dt}\left(\|\nabla c\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \int n\ln n dx\right) + \|\nabla\sqrt{n}\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2\\ \le C\|\nabla c\|_{L^2}^2(\|\nabla u\|_{L^2}^2 + \|\nabla c\|_{L^2}^2).\end{aligned}$$

By Gronwall's inequality, we obtain

$$\begin{aligned}&\left(\|\nabla c\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \int n\ln n dx\right) + \int_0^t \left(\|\nabla\sqrt{n}\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2\right) ds\\ &\le \left(\|\nabla c_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \int_0 n_0\ln n_0 dx\right)\exp\left(C\|c_0\|_{L^2}^2\right).\end{aligned} \tag{2.28}$$

where we used that $\|\nabla c\|_{L^2_{x,t}} \le \|c_0\|_{L^2}$. Next, we estimate $\int n\,|\ln n|\,dx$. For simplicity, we set

$$D_1 = \{x : n(x) \le e^{-|x|}\}, \qquad D_2 = \{x : e^{-|x|} < n(x) \le 1\}.$$

A typical argument for dealing with kinetic entropy (see e.g. [7]), we estimate

$$\left|\int n(\ln n)_-\right| = -\int_{D_1} n\ln n - \int_{D_2} n\ln n \le C\int_{D_1}\sqrt{n} + \int_{D_2}\langle x\rangle n \le C\int e^{-\frac{|x|}{2}} + \int\langle x\rangle n,$$

where $(\ln x)_-$ is a negative part of $\ln x$ and $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$. We compute

$$\frac{d}{dt}\int_{\mathbb{R}^2}\langle x\rangle n dx = \int_{\mathbb{R}^2} nu\nabla\langle x\rangle dx + \int_{\mathbb{R}^2} n\Delta\langle x\rangle dx + \int_{\mathbb{R}^2}\chi(c)n\nabla c\nabla\langle x\rangle dx. \tag{2.29}$$

The term $\int_{\mathbb{R}^2} nu\nabla\langle x\rangle dx$ is estimated by

$$\left|\int_{\mathbb{R}^2} nu\nabla\langle x\rangle dx\right| \le \epsilon\|\nabla\sqrt{n}\|_{L^2}\|u\|_{L^2}.$$

Noting that $|\nabla\langle x\rangle| + |\Delta\langle x\rangle| \le C$, we get

$$\left|\int_{\mathbb{R}^2} n\Delta\langle x\rangle dx\right| + \left|\int_{\mathbb{R}^2} \chi(c)n\nabla c\nabla\langle x\rangle dx\right| \le C\epsilon(1 + \left\|\nabla\sqrt{n}\right\|_{L^2}\|\nabla c\|_{L^2}).$$

Thus, (2.29) is estimated as follows:

$$\frac{d}{dt}\int_{\mathbb{R}^2}\langle x\rangle n dx \le C\epsilon(\|\nabla\sqrt{n}\|_{L^2}^2 + \|u\|_{L^2}^2 + \|\nabla c\|_{L^2}^2 + 1). \tag{2.30}$$

From the u-equation we have

$$\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \le C\epsilon\|\nabla\sqrt{n}\|_{L^2}\|u\|_{L^2},$$

which gives

$$\|u\|_{L^2}^2 + \int_0^t \|\nabla u\|_{L^2}^2 ds \le \|u_0\|_{L^2} + C\epsilon\int_0^t \|\nabla\sqrt{n}\|^2 + \|u\|_{L^2}^2 ds. \tag{2.31}$$

We add $2\left|\int n(\ln n)_- dx\right|$ to (2.28) and using (2.30), (2.31) we then have

$$\begin{aligned}
&\left(\|\nabla c\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \int n|\ln n|dx + \|u\|_{L^2}^2\right) \\
&+ \int_0^t \left(\frac{1}{2}\|\nabla\sqrt{n}\|_{L^2}^2 + \|\Delta c\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\right) ds \\
&\le \left(\|\nabla c_0\|_{L^2}^2 + \|\omega_0\|_{L^2}^2 + \int_0 n_0\ln n_0 dx\right)\exp\left(C\|c_0\|_{L^2}^2\right) \\
&\quad + C + \int\langle x\rangle n_0 dx + C\epsilon\|c_0\|_{L^2}^2 + C\epsilon\|u_0\|_{L^2}^2 t.
\end{aligned}$$

Therefore, we have $\int_0^T \|\nabla\sqrt{n}\|_{L^2}^2 dt \le C(T)$, which implies $n \in L^2_{x,t}$ via (2.25). Therefore, it is direct, due to (1.7) in Theorem 1, that solutions become regular. This completes the proof. $\square$

3 Proof of Theorem 2

In this section we present the proof of Theorem 2. We start with the control of L^p-norm of n under the smallness of $\|c_0\|_{L^\infty}$ in next proposition. We use the similar weighted energy estimate in [22], which treated the case of χ, k are constants and fluid equation is absent. We remark that due to the incompressible condition of u, the proof of [22, Lemma 3.1] can be applicable to our case and the generalization to non-constant $\chi(c), \kappa(c)$ is also available as long as a maximum principle of c holds, i.e. $0 \le c \le \|c_0\|_{L^\infty}$.

Proposition 2 *Let the assumptions in Theorem 1 hold and $p \in (1,\infty)$. There exists $\delta_1 = \delta_1(p)$ such that if $\|c_0\|_{L^\infty} < \delta_1$, then $n(t) \in L^p(\mathbb{R}^d)$ for all $t \in [0, T^*)$ and*

$$\|n(t)\|_{L^p} \le C = C(p, \|c_0\|_{L^\infty}, \|n_0\|_{L^p}), \tag{3.1}$$

where T^* is the maximal time of existence in Theorem 1.

Proof. For a positive $\phi(c)$ such that $\phi'(c) \ge 0$, which will be determined later, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int n^p\phi &= \int n^{p-1}\phi\left(-u\cdot\nabla n + \Delta n - \nabla\cdot(n\chi\nabla c)\right) + \frac{1}{p}\int n^p\phi'\left(-u\cdot\nabla c + \Delta c - nk(c)\right)\\ &= \int n^{p-1}\phi\Delta n - \int n^{p-1}\phi\chi\nabla\cdot(n\nabla c) + \frac{1}{p}\int n^p\phi'\Delta c\\ &\quad - \int n^{p-1}\phi\chi' n|\nabla c|^2 - \frac{1}{p}\int n^p\phi' nk(c) - \int n^{p-1}\phi u\cdot\nabla n - \frac{1}{p}\int n^p\phi' u\cdot\nabla c. \end{aligned}\tag{3.2}$$

We note that, due to $\nabla\cdot u = 0$ such that $\phi' u\cdot\nabla c = \nabla\cdot(\phi u)$, the last two terms in (3.2) are cancelled, i.e. $\int n^{p-1}\phi u\cdot\nabla n + \frac{1}{p}\int n^p\phi' u\cdot\nabla c = 0$. Via the integration by parts, we have

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int n^p\phi + (p-1)\int n^{p-2}\phi|\nabla n|^2 + \frac{1}{p}\int n^p\phi''|\nabla c|^2\\ &= -2\int n^{p-1}\phi'\nabla n\cdot\nabla c + (p-1)\int n^{p-1}\chi\phi\nabla n\cdot\nabla c + \int n^p\chi\phi'|\nabla c|^2 - \frac{1}{p}\int n^p\phi' nk. \end{aligned}\tag{3.3}$$

Noting that the last term in (3.3) is non-positive and using Cauchy-Schwartz inequality,

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int n^p\phi + \frac{p-1}{2}\int n^{p-2}\phi|\nabla n|^2 + \frac{1}{p}\int n^p\phi''|\nabla c|^2\\ &\le \frac{4}{p-1}\int n^p\frac{(\phi')^2}{\phi}|\nabla c|^2 + (p-1)\int n^p\chi^2\phi|\nabla c|^2 + \int n^p\chi\phi'|\nabla c|^2. \end{aligned}\tag{3.4}$$

We set $\phi(c) = e^{(\beta c)^2}$ and we look for ϕ satisfying

$$\frac{4}{p-1}\frac{(\phi')^2}{\phi} + (p-1)\chi^2\phi + \chi\phi' \le \frac{1}{2p}\phi''. \tag{3.5}$$

Let $\chi_1 = \sup_{0\le c\le\|c_0\|_{L^\infty}}\chi(c)$. We then see that (3.5) is satisfied, provided that

$$(p-1)\chi_1^2 \le \frac{1}{3p}\beta^2, \qquad \|c_0\|_{L^\infty}\chi_1 \le \frac{1}{6p}, \qquad \frac{8}{p-1}\beta^2\|c_0\|_{L^\infty}^2 \le \frac{1}{6p}. \tag{3.6}$$

If β is chosen such that $6p(p-1)\chi_1^2 = \beta^2$ and if $\chi_1\|c_0\|_{L^\infty} \le \frac{1}{24p}$, it is straightforward that (3.6) is satisfied. Therefore, if $\|c_0\|_{L^\infty}$ is sufficiently small, we obtain

$$\frac{1}{p}\frac{d}{dt}\int n^p\phi + \frac{p-1}{2}\int n^{p-2}\phi|\nabla n|^2 + \frac{1}{2p}\int n^p\phi''|\nabla c|^2 \le 0.$$

Since $\phi > 1$, it follows that $\int_{\mathbb{R}^d} n^p(t)dx \le e^{\beta^2\|c_0\|_{L^\infty}^2}\int_{\mathbb{R}^d} n_0^p dx$. This completes the proof. $\square$

We remark that Proposition 2 dose not give control of $\|n\|_{L^\infty}$. Toward the boundedness as well as the decay of $\|n\|_{L^\infty}$, we modify the approach done in [20], where the degenerate Keler-Segel system (1.12) was considered. With the aid of incompressibility of the velocity vector field u, it turns out that the method of proof in [20] can be adjusted properly to our case. Now we are ready to present the proof of Theorem 2.

Proof of Theorem 2 We first note that solutions are classical, because of Theorem 1 and Proposition 2. To obtain a truncated energy inequality for (P-KSNS), we first differentiate $\int (n-K)_+^p \phi$ in time variable, where ϕ is the function introduced in the proof of Proposition 2. Similarly as in (3.2), (3.4), we have

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int (n-K)_+^p \phi + (p-1)\int (n-K)_+^{p-2}\phi|\nabla(n-K)_+|^2 + \frac{1}{p}\int (n-K)_+^p \phi''|\nabla c|^2 \\
&= -2\int (n-K)_+^{p-1}\phi'\nabla(n-K)_+\cdot\nabla c - \underbrace{\frac{1}{p}\int (n-K)_+^p \phi' n\kappa}_{\geq 0} \\
&+ (p-1)\int (n-K)_+^{p-1}\chi\phi\nabla(n-K)_+\cdot\nabla c + \int (n-K)_+^p \chi\phi'|\nabla c|^2 \\
&+ K(p-1)\int (n-K)_+^{p-2}\chi\phi\nabla(n-K)_+\cdot\nabla c + K\int (n-K)_+^{p-1}\chi\phi'|\nabla c|^2.
\end{aligned}$$

We note that the last two integrands in the equality above are bounded as follows:

$$(n-K)_+^{p-2}\phi\chi\nabla(n-K)_+\cdot\nabla c \leq \frac{1}{4K}(n-K)_+^{p-2}\phi|\nabla(n-K)_+|^2 + 4K(n-K)_+^{p-2}\phi\chi^2|\nabla c|^2,$$

$$(n-K)_+^{p-1}\chi\phi'|\nabla c|^2 \leq \left(\frac{1}{8K}(n-K)_+^p + 8K\right)\chi\phi'|\nabla c|^2.$$

In what follows, we fix $p=2$. With the aid of (3.6), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (n-K)_+^2\phi + \frac{1}{4}\int \phi|\nabla(n-K)_+|^2 + \frac{1}{8}\int (n-K)_+^2\phi''|\nabla c|^2 \\
&\leq 4K^2\int \phi\chi^2|\nabla c|^2 + 8K^2\int \chi\phi'|\nabla c|^2 \leq 8K^2\int (\phi\chi^2+\chi\phi')|\nabla c|^2.
\end{aligned} \tag{3.7}$$

We set $L := \sup_{0\leq c\leq \|c_0\|_{L^\infty}} \big(\phi\chi^2+\chi\phi'\big)$. Multiplying the equation c with $16K^2L$, we have

$$\frac{d}{dt}\int 8K^2L(c-K)_+^2 + \int 16K^2L|\nabla c|^2 \leq 0. \tag{3.8}$$

Summing up (3.7) and (3.8), we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int \frac{1}{2}(n-K)_+^2\phi + \int 8K^2L(c-K)_+^2\right) + \frac{1}{4}\int \phi|\nabla(n-K)_+|^2 \\
&+ \frac{1}{8}\int (n-K)_+^2\phi''|\nabla c|^2 + 8K^2L\int |\nabla c|^2 \leq 0.
\end{aligned} \tag{3.9}$$

Similarly proceeding as in [20], we define

$$U(\xi) = \int_0^T \nu(t)\int (n-\xi\eta(t))_+^2 + \int_0^T \nu(t)\int (c-\xi\eta(t))_+^2 := U_1(\xi)+U_2(\xi), \quad \xi>0.$$

Here the auxiliary functions $\nu(t), \eta(t)$, and the range of $\xi \in [0, 2M]$ are specified later, where $\nu(t)$ and $\eta(t)$ are decreasing in t and M is a fixed number. We then note that

$$U'(\xi) = -2\int_0^T \nu(t)\eta(t)\int (n-\xi\eta(t))_+ - 2\int_0^T \nu(t)\eta(t)\int (c-\xi\eta(t))_+. \tag{3.10}$$

Let $K_1 := \sup_{0\le\xi\le 2M, t>0} \xi\eta(t)$. Repeating similar computations as in (3.9), we observe that

$$\begin{aligned}
&\frac{d}{dt}\left(\int \frac{1}{2}(n-\xi\eta(t))_+^2\phi + 8K_1^2 L\int (c-\xi\eta(t))_+^2\right) + \frac{1}{4}\int \phi|\nabla(n-\xi\eta(t))_+|^2 \\
&+\frac{1}{8}\int (n-\xi\eta(t))_+^2\phi''|\nabla c|^2 + 8K_1^2 L\int |\nabla(c-\xi\eta(t))_+|^2 \\
&\le -\xi\eta'(t)\int (n-\xi\eta(t))_+\phi dx - \xi\eta'(t)\int (c-\xi\eta(t))_+.
\end{aligned} \tag{3.11}$$

We set $\phi_{\min} = \min\phi(c) \ge 1$. For simplicity, we define

$$\begin{aligned}
E(\xi) := \sup_{0\le t\le T}&\left(\int \frac{1}{2}(n-\xi\eta(t))_+^2 + \frac{8K_1^2L}{\phi_{\min}}\int \frac{1}{2}(c-\xi\eta(t))_+^2\right) + \frac{1}{4}\int_0^T\int |\nabla(n-\xi\eta(t))_+|^2 \\
&+\frac{1}{8}\frac{1}{\phi_{\min}}\int_0^T\int (n-\xi\eta(t))_+^2\phi''|\nabla c|^2 + \frac{8K_1^2L}{\phi_{\min}}\int_0^T\int |\nabla(c-\xi\eta(t))_+|^2.
\end{aligned}$$

Then, after integrating (3.11) in time variable for $\xi \ge \xi_0 := \eta^{-1}(0)\max\{\|n_0\|_{L^\infty}, \|c_0\|_{L^\infty}\}$, we obtain

$$E(\xi) \le -\frac{1}{\phi_{\min}}\left(\int_0^T \xi\eta'(t)\int (n-\xi\eta(t))_+\phi dx + \int_0^T \xi\eta'(t)\int (c-\xi\eta(t))_+\right). \tag{3.12}$$

Assuming that $|\eta'(t)| \le C\nu(t)\eta(t)$, which will be confirmed later, we have

$$E(\xi) \le C\xi|U'(\xi)|. \tag{3.13}$$

We use the Sobolev embedding and then by interpolating we have

$$\|(n-\xi\eta(t))_+\|_{L^q_{t,x}}^2 + \|(c-\xi\eta(t))_+\|_{L^q_{t,x}}^2 \le CE(\xi), \qquad q = 2(d+2)/d, \quad d\ge 2. \tag{3.14}$$

For simplicity, we denote

$$A := \int_0^T\int \nu(t)^{\frac{1}{\alpha}}(n-\xi\eta(t))_+, \qquad B := \int_0^T\int \nu(t)^{\frac{1}{\alpha}}(c-\xi\eta(t))_+.$$

Interpolating $1 < 2 < q$ in space and using the Hölder's inequality in time, we have

$$U_1(\xi) \le A^\alpha E^\theta(\xi), \qquad U_2(\xi) \le B^\alpha E^\theta(\xi), \qquad \alpha = \frac{4}{4+d}, \ \theta = \frac{d+2}{d+4}. \tag{3.15}$$

Under $\nu(t)\eta(t) \ge C\nu(t)^{\frac{1}{\alpha}} = C\nu(t)^{1+\frac{d}{4}}$, we obtain from (3.10) together with (3.13) and (3.15)

$$|U'(\xi)| \ge CA + CB \ge CU^{\frac{1}{\alpha}}E^{-\theta/\alpha} \ge C\xi^{-\theta/\alpha}U^{\frac{1}{\alpha}}|U'|^{-\theta/\alpha}, \tag{3.16}$$

where we used that $(A+B)^\alpha \geq C_\alpha(A^\alpha + B^\alpha)$ for $0<\alpha<1$. Now we choose the auxillary functions ν and η by

$$\nu(t) = (1+t)^{-1}, \qquad \eta(t) = (1+t)^{-\frac{d}{4}}.$$

By the similar reasoning mentioned in [20], we need to show that $U(\xi)$ is finite for some $\xi>0$. Indeed, for $q = 2(d+2)/d$ we have

$$\begin{aligned} U(2\xi) &= \int_0^T \int_{\{n\geq 2\xi\eta(t)\}} \nu(t)\,(n-2\xi\eta(t))_+^2\,dxdt + \int_0^T \int_{\{n\geq 2\xi\eta(t)\}} \nu(t)\,(c-2\xi\eta(t))_+^2\,dxdt \\ &\leq \int_0^T \nu(t) \int_{\{n\geq 2\xi\eta(t)\}} (n-\xi\eta(t))_+^{\frac{2(d+2)}{d}} (\xi\eta(t))^{-\frac{4}{d}} dxdt \\ &\quad + \int_0^T \nu(t) \int_{\{n\geq 2\xi\eta(t)\}} (c-\xi\eta(t))_+^{\frac{2(d+2)}{d}} (\xi\eta(t))^{-\frac{4}{d}} dxdt, \end{aligned}$$

where we used that $n-\xi\eta(t) \geq \xi\eta(t)$ on $\{x|n-2\xi\eta(t)\geq 0\}$. Therefore, we obtain

$$\begin{aligned} U(2\xi) &\leq \int_0^T \frac{\nu(t)}{(\xi\eta(t))^{\frac{4}{d}}} \int n^{\frac{2(d+2)}{d}} dxdt + \int_0^T \frac{\nu(t)}{(\xi\eta(t))^{\frac{4}{d}}} \int c^{\frac{2(d+2)}{d}} dxdt \\ &\leq C\xi^{-\frac{4}{d}} \left(\int_0^T \|n\|_{\frac{2(d+2)}{d}}^{\frac{2(d+2)}{d}} dt + \int_0^T \|c\|_{\frac{2(d+2)}{d}}^{\frac{2(d+2)}{d}} dt \right) \\ &\leq C\xi^{-\frac{4}{d}} (\|n_0\|_2^{\frac{2(d+2)}{d}} + \|c_0\|_2^{\frac{2(d+2)}{d}}), \end{aligned} \tag{3.17}$$

where the last inequality in (3.17) is due to the case of $K=0$ in (3.9) together with (3.8) and (3.14). Therefore, (3.17) implies that $U(\xi)$ is finite for every $\xi>0$ as long as $U(\xi)$ exists. Via (3.16) and (3.17), we observe that

$$U'(\xi) \leq -C\xi^{-\frac{d+2}{d+6}} U^{\frac{d+4}{d+6}}, \qquad \xi > \xi_0 := \eta^{-1}(0)\{\|n_0\|_{L^\infty}, \|c_0\|_{L^\infty}\}.$$

and it is immediate that $U(\xi)$ vanish at a finite value $\xi = M(\xi_0, \|n_0\|_{L^2}, \|c_0\|_{L^2})$. Summing up the arguments, we conclude that $n(x,t)+c(x,t) \leq CM(1+t)^{-\frac{d}{4}}$ for $t>0$. This completes the proof. □

Remark 3 *The L^2 energy inequality* (3.9) *is responsible for the time decay rate $t^{-d/4}$. The number coincides to that for the solution of the heat equation with the initial data in $L^2(\mathbb{R}^d)$. We do not know whether or not such decay estimate can be improved, and thus we leave it an open question.*

4 Blow up criteria of parabolic-hyperbolic system

In this section, we consider (1.11), which is the case that equation of c is of no diffusion. First we construct solutions of (1.11) locally in time in the following class of functions:

$$X_T^s := (C([0,T);\, H^s) \cap L^2(0,T;H^{s+1}) \times C([0,T);\, H^{s+1}) \times (C([0,T);\, H^s) \cap L^2(0,T;H^{s+1}).$$

Our construction of regular solutions is based on the method of contraction mapping via linearizing the equations in an iterative way. Next proposition is the first part of Theorem 3.

Proposition 3 *Let initial data, χ, k, and ϕ satisfy the assumptions in Theorem 3. Then there exists $T > 0$ depending on $\|n_0\|_{H^s}, \|c_0\|_{H^{s+1}}, \|u_0\|_{H^s}$ with integer $s > 2$ such that a unique solution (n, c, u) in X^s_T exists.*

Proof. We consider following linearized system, which is defined iteratively (set $(n^0, c^0, u^0) = (n_0, c_0, u_0)$) over $\mathbb{R}^d \times (0, T)$ with $d = 2, 3$.

$$\begin{cases} \partial_t n^{(m+1)} + (u^{(m)} \cdot \nabla) n^{(m+1)} - \Delta n^{(m+1)} = -\nabla \cdot [\chi(c^{(m)}) n^{(m)} \nabla c^{(m)}], \\ \partial_t c^{(m+1)} + (u^{(m)} \cdot \nabla) c^{(m+1)} = -k(c^{(m)}) n^{(m)}, \\ \partial_t u^{(m+1)} + (u^{(m)} \cdot \nabla) u^{(m+1)} - \Delta u^{(m+1)} + \nabla p^{(m+1)} = n^{(m)} \nabla \phi, \\ \text{div } u^{(m+1)} = 0. \end{cases} \tag{4.1}$$

• (Uniform boundedness) If the initial data $(n_0, c_0, u_0) \in H^s \times H^{s+1} \times H^s$ with integer $s > \left[\frac{d}{2}\right] + 1$, then we show $(n^{(m)}, c^{(m)}, u^{(m)})$ is uniformly bounded in $X^s_{T_0}$ for some $T_0 > 0$. Let $\alpha = (\alpha_1, \cdots, \alpha_d)$ be a multi-index and $|\alpha| := \alpha_1 + \cdots + \alpha_d$. Taking D^α operator on the first equation in (4.1), taking scalar product with $D^\alpha n^{(m+1)}$ and summing over $|\alpha| \le s$, we have

$$\frac{1}{2}\frac{d}{dt}\|n^{(m+1)}\|^2_{H^s} + \|\nabla n^{(m+1)}\|^2_{H^s} \le -\sum_{|\alpha|\le s} \int_{\mathbb{R}^d} D^\alpha(u^{(m)} \cdot \nabla n^{(m+1)}) \cdot D^\alpha n^{(m+1)} dx$$

$$+ \sum_{|\alpha|\le s} \int_{\mathbb{R}^d} D^\alpha(\chi(c^{(m)}) n^{(m)} \nabla c^{(m)}) \cdot D^\alpha \nabla n^{(m+1)} dx := I_1 + I_2.$$

Using cancellation and calculus inequality, we obtain

$$|I_1| \le \sum_{|\alpha|\le s} \left| \int_{\mathbb{R}^d} (D^\alpha(u^{(m)} \cdot \nabla n^{(m+1)}) - u^{(m)} \cdot \nabla D^\alpha n^{(m+1)}) \cdot D^\alpha n^{(m+1)} dx \right|$$

$$\le C(\|\nabla u^{(m)}\|_{L^\infty} \|n^{(m+1)}\|_{H^s} + \|u^{(m)}\|_{H^s} \|\nabla n^{(m+1)}\|_{L^\infty}) \|n^{(m+1)}\|_{H^s} \le C\|u^{(m)}\|_{H^s} \|n^{(m+1)}\|^2_{H^s}.$$

Using Young's inequality and interpolation inequality, we have

$$|I_2| \le C \sum_{|\alpha|\le s} \|D^\alpha(\chi(c^{(m)}) n^{(m)} \nabla c^{(m)})\|^2_{L^2} + \frac{1}{2}\|\nabla n^{(m+1)}\|^2_{H^s}$$

$$\le C(1 + \|c^{(m)}\|^{2s}_{H^{s+1}}) \|n^{(m)}\|^2_{H^s} \|c^{(m)}\|^2_{H^{s+1}} + \frac{1}{2}\|\nabla n^{(m+1)}\|^2_{H^s}.$$

We find that

$$\frac{d}{dt}\|n^{(m+1)}\|^2_{H^s} + \|\nabla n^{(m+1)}\|^2_{H^s} \le C\|u^{(m)}\|_{H^s} \|n^{(m+1)}\|^2_{H^s}$$
$$+ C(1 + \|c^{(m)}\|^{2s}_{H^{s+1}}) \|n^{(m)}\|^2_{H^s} \|c^{(m)}\|^2_{H^{s+1}}.$$

Similarly, we have

$$\frac{d}{dt}\|c^{(m+1)}\|^2_{H^{s+1}} \le C\|\nabla u^{(m)}\|_{H^s} \|c^{(m+1)}\|^2_{H^{s+1}}$$
$$+ C(\|\nabla n^{(m)}\|_{H^s} + \|n^{(m)}\|_{L^\infty})(1 + \|c^{(m)}\|^s_{H^{s+1}}) \|c^{(m+1)}\|_{H^{s+1}},$$

and

$$\frac{d}{dt}\|u^{(m+1)}\|^2_{H^s} + \|\nabla u^{(m+1)}\|^2_{H^s} \le C\|u^{(m)}\|_{H^s} \|u^{(m+1)}\|^2_{H^s} + C\|n^{(m)}\|_{H^s} \|u^{(m+1)}\|_{H^s}.$$

Adding above, we have

$$\frac{d}{dt}(\|n^{(m+1)}\|_{H^s}^2 + \|c^{(m+1)}\|_{H^{s+1}}^2 + \|u^{(m+1)}\|_{H^s}^2) + \|\nabla n^{(m+1)}\|_{H^s}^2 + \|\nabla u^{(m+1)}\|_{H^s}^2$$

$$\le C(\|u^{(m)}\|_{H^s} + \|\nabla u^{(m)}\|_{H^s} + \|\nabla n^{(m)}\|_{H^s} + \|n^{(m)}\|_{L^\infty} + 1)\times$$
$$(\|n^{(m+1)}\|_{H^s}^2 + \|c^{(m+1)}\|_{H^{s+1}}^2 + \|u^{(m+1)}\|_{H^s}^2) + C(1 + \|c^{(m)}\|_{H^{s+1}}^{2s})\|n^{(m)}\|_{H^s}^2\|c^{(m)}\|_{H^{s+1}}^2$$
$$+C(\|\nabla n^{(m)}\|_{H^s} + \|n^{(m)}\|_{L^\infty})(1 + \|c^{(m)}\|_{H^{s+1}}^s)^2 + C\|n^{(m)}\|_{H^s}^2.$$

Gronwall's inequality gives uniform boundedness in $X_{T_0}^s$ for some $T_0 > 0$, because

$$\sup(\|n^{(m+1)}\|_{H^s}^2 + \|c^{(m+1)}\|_{H^{s+1}}^2 + \|u^{(m+1)}\|_{H^s}^2) + \int_0^{T_0} \|\nabla n^{(m+1)}\|_{H^s}^2 + \|\nabla u^{(m+1)}\|_{H^s}^2 dt$$

$$\le \left(\|n_0\|_{H^s}^2 + \|c_0\|_{H^{s+1}}^2 + \|u_0\|_{H^s}^2 + CT_0^{1/2}(M^{1/2} + M^{s+2})\right)\exp\left(CT_0^{1/2}(1 + M^{1/2})\right),$$

under the hypothesis

$$\sup_{t\in[0,T_0]}(\|n^{(m)}\|_{H^s}^2 + \|c^{(m)}\|_{H^{s+1}}^2 + \|u^{(m)}\|_{H^s}^2) + \int_0^{T_0} \|\nabla n^{(m)}\|_{H^s}^2 + \|\nabla u^{(m)}\|_{H^s}^2 dt \le M.$$

- (Convergence) To show that $\{(n^{(m)}, c^{(m)}, u^{(m)})\}$ is a Cauchy sequence in X_T^s for some $0 < T_1 < T_0$, we consider the equations of the difference of solutions

$$\begin{cases} \partial_t(n^{(m+1)} - n^{(m)}) - \Delta(n^{(m+1)} - n^{(m)}) + (u^{(m)}\cdot\nabla)(n^{(m+1)} - n^{(m)}) + (u^{(m)} - u^{(m-1)})\nabla n^{(m)} \\ \qquad = -\nabla\cdot[\chi(c^{(m)})n^{(m)}\nabla c^{(m)}] + \nabla\cdot[\chi(c^{(m-1)})n^{(m-1)}\nabla c^{(m-1)}], \\ \partial_t(c^{(m+1)} - c^{(m)}) + (u^{(m)}\cdot\nabla)(c^{(m+1)} - c^{(m)}) + (u^{(m)} - u^{(m-1)})\cdot\nabla c^{(m)} \\ \qquad = -k(c^{(m)})n^{(m)} + k(c^{(m-1)})n^{(m-1)}, \\ \partial_t(u^{(m+1)} - u^{(m)}) - \Delta(u^{(m+1)} - u^{(m)}) + (u^{(m)}\cdot\nabla)(u^{(m+1)} - u^{(m)}) \\ \qquad +(u^{(m)} - u^{(m-1)})\cdot\nabla u^{(m)} + \nabla(p^{(m+1)} - p^{(m)}) = (n^{(m)} - n^{(m-1)})\nabla\phi, \\ \mathrm{div}\,(u^{(m+1)} - u^{(m)}) = 0. \end{cases}$$

Following the arguments similarly in [1], we can prove the convergence. Since its verification is rather straightforward, the details are omitted. □

To obtain the blow-up criteria in Theorem 3, we derive lengthy a priori estimates. Especially, the estimates of $\|\nabla u\|_{L_t^2 L_x^\infty}$ is crucial. To obtain a bound of $\|\nabla u\|_{L_t^2 L_x^\infty}$, we first use vorticity estimates to obtain L^2 estimates of $\nabla\omega$, and then, we obtain the estimates $\|\nabla u\|_{L_{t,x}^{2,\infty}}$ by using the mixed norms $L_{t,x}^{q,p}$ type estimates for Stokes system (see e.g. [10]). Then the desired blow-up criterion can be obtained by an induction argument. This is the outline of the second part of Theorem 3 and now we give the proof.

Proof of Theorem 3 Since construction of local solution is done in Proposition 3, it remains to show the blow-up criteria for $(2D)$ and $(3D)$. We will show those criteria by obtaining a priori estimates as in the below steps for $[0,T]$ for any $T < T^*$, where T^* is the maximal time of existence. Since $\int_0^T \|n\|_{L^\infty}^2 dt < \infty$ and $\|n(t)\|_{L^1} = \|n_0\|_{L^1}$ for all $t \in [0,T]$, we note that $\int_0^T \|n\|_{L^p}^2 dt < \infty$ for all $p < \infty$.

• (Case $\mathbb{R}^2$) At first, we consider the case $d = 2$.
Step 1-1 ($L^2 \times H^1 \times H^1$ Estimates of (n, c, u)). Testing u to the equation of u, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \le C\|n\|_{L^2}\|u\|_{L^2} \le \|n\|_{L^2}^2 + C\|u\|_{L^2}^2.$$

It follows from integration in time that

$$\sup_{0<t\le T}\|u(t)\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2 dt \le C(\|u_0\|_{L^2}^2 + \int_0^T \|n\|_{L^2}^2 dt). \tag{4.2}$$

Consider the equation of the vorticity $\omega = \partial_1 u_2 - \partial_2 u_1$.

$$\partial_t \omega + (u \cdot \nabla)\omega - \Delta\omega = \nabla \times (n\nabla\phi). \tag{4.3}$$

Multiplying (4.3) with ω and integrating, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2 + \|\nabla\omega\|_{L^2}^2 \le \left|\int_{\mathbb{R}^2} n\nabla\phi \times \nabla\omega dt\right| \le C\|n\|_{L^2}\|\nabla\omega\|_{L^2} \le C\|n\|_{L^2}^2 + \frac{1}{2}\|\nabla\omega\|_{L^2}^2,$$

and, therefore, we obtain

$$\sup_{0<t\le T}\|\omega(t)\|_{L^2}^2 + \int_0^T \|\nabla\omega\|_{L^2}^2 dt \le \|\omega_0\|_{L^2}^2 + \int_0^T \|n\|_{L^2}^2 dt.$$

On the other hand, due to mixed norm estimate of Stokes system (see e.g. [10]), we note that for any $p \in (1, \infty)$

$$\int_0^T \|\Delta u\|_{L^p}^2 dt \le C\|u_0\|_{H^2}^2 + C\int_0^T \|n(t)\|_{L^p}^2 dt + C\int_0^T \|u\|_{L^{2p}}^2\|\nabla u\|_{L^{2p}}^2 dt < \infty, \tag{4.4}$$

where we used

$$\int_0^T \|(u \cdot \nabla)u\|_{L^p}^2 dt \le \int_0^T \|u\|_{L^{2p}}^2\|\nabla u\|_{L^{2p}}^2 dt \le \|u\|_{L^\infty(0,T;L^{2p})}^2\|\nabla u\|_{L^2(0,T;L^{2p})}^2.$$

Hence it follows that $\|\nabla u\|_{L^2(0,T;L^\infty)} < \infty$. Next, testing n to the equation of n, we obtain

$$\frac{1}{2}\frac{d}{dt}\|n(t)\|_{L^2}^2 + \|\nabla n\|_{L^2}^2 \le \int_{\mathbb{R}^2} \chi(c)n\nabla c\nabla n dx \le \frac{1}{4}\|\nabla n\|_{L^2}^2 + C\|n\|_{L^\infty}^2\|\nabla c\|_{L^2}^2. \tag{4.5}$$

Taking ∇ on the equation of c, multiplying ∇c and integrating over $\mathbb{R}^2$ yield that

$$\frac{1}{2}\frac{d}{dt}\|\nabla c\|_{L^2}^2 \le \left|\int_{\mathbb{R}^2} \nabla u\nabla c\nabla c dx\right| + \left|\int_{\mathbb{R}^2} \nabla(k(c)n)\nabla c dx\right|$$

$$\le C\|\nabla u\|_{L^\infty}\|\nabla c\|_{L^2}^2 + C\|n\|_{L^\infty}\|\nabla c\|_{L^2}^2 + C\|n\|_{L^\infty}\|\nabla c\|_{L^2}^2 + \frac{1}{4}\|\nabla n\|_{L^2}^2. \tag{4.6}$$

If we add the above two inequalities (4.5) and (4.6), then we have

$$\frac{d}{dt}(\|n(t)\|_{L^2}^2 + \|\nabla c(t)\|_{L^2}^2) + \|\nabla n\|_{L^2}^2 \le C(\|\nabla u\|_{L^\infty} + C\|n\|_{L^\infty}^2 + 1)(\|n\|_{L^2}^2 + \|\nabla c\|_{L^2}^2). \tag{4.7}$$

Using Gronwall's Lemma, we have

$$\sup_{t\in(0,T]}(\|n(t)\|_{L^2}^2+\|\nabla c(t)\|_{L^2}^2)+\int_0^T\|\nabla n(t)\|_{L^2}^2dt\le C(\|n_0\|_{L^2}^2+\|\nabla c_0\|_{L^2}^2)$$

$$\times\exp(C(T^{1/2}\|\nabla u\|_{L^2(0,T;L^\infty)}+\|n\|_{L^2(0,T;L^\infty)}^2+T))\int_0^T(\|\nabla u\|_{L^\infty}+C\|n\|_{L^\infty}^2+1)dt<\infty.$$

Step 1-2 (Induction argument) Assuming that for an integer m with $1\le m$

$$n\in L^\infty(0,T;H^{m-1})\cap L^2(0,T;H^m),\quad c\in L^\infty(0,T:H^m) \tag{4.8}$$

and

$$u\in L^\infty(0,T;H^m)\cap L^2(0,T;H^{m+1}). \tag{4.9}$$

we will show that

$$n\in L^\infty(0,T;H^m)\cap L^2(0,T;H^{m+1}),\qquad c\in L^\infty(0,T:H^{m+1}),$$

$$u\in L^\infty(0,T;H^{m+1})\cap L^2(0,T;H^{m+2}).$$

First, we take D^α operator ($\alpha=(\alpha_1,\alpha_2)$ is a multi index satisfying $|\alpha|=\alpha_1+\alpha_2$, $|\alpha|\le m+1$) with the equations of u, scalar product them with $D^\alpha u$ and sum over $|\alpha|\le m+1$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^{m+1}}^2+\|\nabla u\|_{H^{m+1}}^2\le-\sum_{|\alpha|\le m+1}\int_{\mathbb{R}^2}D^\alpha((u\cdot\nabla)u)D^\alpha udx+C\|n\|_{H^m}\|\nabla u\|_{H^{m+1}}.$$

If we use the commutator estimates such that

$$\left|\sum_{|\alpha|=0}^{m+1}\int D^\alpha((u\cdot\nabla)u)D^\alpha u\right|=\left|\sum_{|\alpha|=0}^{m+1}\int[D^\alpha((u\cdot\nabla)u)-(u\cdot\nabla)D^\alpha u]D^\alpha u\right|\le C\|\nabla u\|_{L^\infty}\|u\|_{H^{m+1}}^2,$$

then we have

$$\frac{d}{dt}\|u\|_{H^{m+1}}^2+\|\nabla u\|_{H^{m+1}}^2\le C\|\nabla u\|_{L^\infty}\|u\|_{H^{m+1}}^2+C\|n\|_{H^m}^2.$$

Gronwall's inequality gives us that

$$u\in L^\infty(0,T;H^{m+1})\cap L^2(0,T;H^{m+2}).$$

Next, we take D^α operator ($\alpha=(\alpha_1,\alpha_2)$ is a multi index satisfying $|\alpha|=\alpha_1+\alpha_2$, $|\alpha|\le m$) with the equations of n , scalar product them with $D^\alpha n$ and sum over $|\alpha|\le m$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|n\|_{H^m}^2+\|\nabla n\|_{H^m}^2\le-\sum_{|\alpha|\le m}\int_{\mathbb{R}^2}D^\alpha((u\cdot\nabla)n)D^\alpha ndx+C\|\chi(c)n\nabla c\|_{H^m}\|\nabla n\|_{H^m}.$$

Using integration by parts (choose $\alpha_j\ne0$) and calculus inequality, we have

$$\left|\sum_{|\alpha|\le m}\int_{\mathbb{R}^2}D^\alpha((u\cdot\nabla)n)D^\alpha ndx\right|=\left|\sum_{|\alpha|\le m}\int_{\mathbb{R}^3}D^{\alpha-e_j}((u\cdot\nabla)n)D^{\alpha+e_j}ndx\right|$$

$$\le C\|u\cdot\nabla n\|_{H^{m-1}}\|\nabla n\|_{H^m} \le C(|u|_{L^\infty}\|n\|_{H^m} + C\|u\|_{W^{m-1,\infty}}\|\nabla n\|_{L^2})\|\nabla n\|_{H^m}$$
$$\le C(\|u\|_{L^\infty}^2 + \|u\|_{W^{m-1,\infty}}^2)\|n\|_{H^m}^2 + \frac{1}{6}\|\nabla n\|_{H^m}^2. \tag{4.10}$$

Also we have

$$\|\chi(c)n\nabla c\|_{H^m} \le C\|\nabla c\|_{L^4}\|n\|_{W^{m,4}} + C\|n\|_{L^\infty}\|\chi(c)\nabla c\|_{H^m}$$
$$\le C\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{H^1}^{\frac{1}{2}}\|n\|_{H^m}^{\frac{1}{2}}\|\nabla n\|_{H^m}^{\frac{1}{2}} + C_1\|n\|_{L^\infty}(\|\nabla c\|_{H^m} + C_2), \tag{4.11}$$

where C_1 and C_2 are absolute constants depending only on $\|c\|_{H^m}$, which is bounded in the inductive assumption $(m-1)$-th step. Using (4.10) and (4.11), we have

$$\frac{1}{2}\frac{d}{dt}\|n\|_{H^m}^2 + \|\nabla n\|_{H^m}^2 \le C(\|u\|_{L^\infty}^2 + \|u\|_{W^{m-1,\infty}}^2)\|n\|_{H^m}^2$$
$$+ C(\|\nabla c\|_{L^2}^2\|n\|_{H^m}^2 + \|n\|_{L^\infty}^2)\|\nabla c\|_{H^m} + C\|n\|_{L^\infty}^2 + \frac{1}{3}\|\nabla n\|_{H^m}^2. \tag{4.12}$$

Similarly, taking H^{m+1} scalar product equation of c with $D^\alpha c$ and summing over $|\alpha| \le m+1$,

$$\frac{1}{2}\frac{d}{dt}\|c\|_{H^{m+1}}^2 \le -\sum_{|\alpha|\le m+1}\int_{\mathbb{R}^2} D^\alpha((u\cdot\nabla)c)D^\alpha c dx + C\|k(c)n\|_{H^{m+1}}\|c\|_{H^{m+1}}.$$

Using commutator estimates

$$\left|\sum_{|\alpha|\le m+1}\int_{\mathbb{R}^2} D^\alpha((u\cdot\nabla)c)D^\alpha c dx\right| = \left|\sum_{|\alpha|\le m+1}\int_{\mathbb{R}^2}[D^\alpha((u\cdot\nabla)c) - (u\cdot\nabla)D^\alpha c]D^\alpha c dx\right|$$
$$\le C\|\nabla u\|_{L^\infty}\|c\|_{H^{m+1}}^2 + C\|\nabla c\|_{L^4}\|u\|_{W^{m+1,4}}\|c\|_{H^{m+1}}$$
$$\le C\|\nabla u\|_{L^\infty}\|c\|_{H^{m+1}}^2 + C\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{H^1}^{\frac{1}{2}}\|u\|_{H^{m+1}}^{\frac{1}{2}}\|\nabla u\|_{H^{m+1}}^{\frac{1}{2}}\|c\|_{H^{m+1}} \tag{4.13}$$

and Leipniz formula

$$\|k(c)n\|_{H^{m+1}} \le C\|k(c)\|_{H^{m+1}}\|n\|_{L^\infty} + C\|k(c)\|_{L^\infty}\|\nabla n\|_{H^{m+1}}$$
$$\le (C_1\|c\|_{H^{m+1}} + C_2)\|n\|_{L^\infty} + C\|\nabla n\|_{H^{m+1}},$$

where C_1 and C_2 are absolute constants depending only on $\|c\|_{H^m}$ bounded in the inductive assumption $(m-1)$-th step, we have

$$\frac{1}{2}\frac{d}{dt}\|c\|_{H^{m+1}}^2 \le C(\|\nabla u\|_{L^\infty} + \|n\|_{L^\infty} + 1)\|c\|_{H^{m+1}}^2 + C\|n\|_{L^\infty}^2$$
$$+ C\|\nabla c\|_{L^2}^2\|u\|_{H^{m+1}}^2\|\nabla u\|_{H^{m+1}}^2 + \frac{1}{6}\|\nabla n\|_{H^{m+1}}. \tag{4.14}$$

Adding (4.12) and (4.14), we have

$$\frac{d}{dt}(\|n\|_{H^m}^2 + \|c\|_{H^{m+1}}^2) + \|\nabla n\|_{H^{m+1}}^2$$
$$\le C(\|\nabla c\|_{L^2}^2\|n\|_{H^m}^2 + \|n\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty} + \|n\|_{L^\infty} + 1)(\|n\|_{H^m}^2 + \|c\|_{H^{m+1}}^2)$$
$$+C\|n\|_{L^\infty}^2 + C\|\nabla c\|_{L^2}^2\|u\|_{H^{m+1}}^2\|\nabla u\|_{H^{m+1}}^2.$$

Since

$$\|\nabla c\|^2_{L^\infty(0,T;L^2)}\|n\|^2_{L^2(0,T;H^m)}+\|n\|^2_{L^2(0,T;L^\infty)}$$
$$+\|\nabla u\|_{L^1(0,T;L^\infty)}+\|\nabla c\|^2_{L^\infty(0,T;L^2)}\|u\|^2_{L^\infty(0,T;H^{m+1})}\|\nabla u\|^2_{L^2(0,T;H^{m+1})}<\infty,$$

it follows via Gronwall's inequality that

$$\|n\|_{L^\infty(0,T;H^m)}+\|c\|_{L^\infty(0,T;H^{m+1})}+\|\nabla n\|_{L^2(0,T;H^m)}<\infty.$$

This completes the proof of 2D case.

• (Case $\mathbb{R}^3$) Next, we consider the case $d=3$.
Step 2-1 ($L^2\times H^1\times H^1$ Estimates of (n,c,u)). Following similar computations as in 2D case, we also have the estimate (4.2). We recall the equation of the vorticity $\omega=\nabla\times\omega$

$$\partial_t\omega+(u\cdot\nabla)\omega-\Delta\omega=(\omega\cdot\nabla)u+\nabla\times(n\nabla\phi). \tag{4.15}$$

Multiplying (4.3) with ω and integrating in spatial variables, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega\|^2_{L^2}+\|\nabla\omega\|^2_{L^2}\le\left|\int_{\mathbb{R}^2}n\nabla\phi\times\nabla\omega dt\right|+\left|\int_{\mathbb{R}^2}|u\omega||\nabla\omega|dx\right|$$
$$\le C\|n\|_{L^2}\|\nabla\omega\|_{L^2}+\|u\omega\|_{L^2}\|\nabla\omega\|_{L^2}\le C\|n\|^2_{L^2}+C\|u\omega\|^2_{L^2}+\frac{1}{4}\|\nabla\omega\|^2_{L^2}$$
$$\le C\|u\|^2_{L^\beta}\|\omega\|_{L^{\frac{2\beta}{\beta-2}}}+C\|n\|^2_{L^2}+\frac{1}{4}\|\nabla\omega\|^2_{L^2}\le C\|u\|^{\frac{2\beta}{2\beta-3}}_{L^\beta}\|\omega\|^2_{L^2}+C\|n\|^2_{L^2}+\frac{1}{2}\|\nabla\omega\|^2_{L^2}$$

and, therefore, we have

$$\sup_{0<t\le T}\|\omega(t)\|^2_{L^2}+\int_0^T\|\nabla\omega\|^2_{L^2}dt\le\left(\|\omega_0\|^2_{L^2}+C\int_0^T\|n\|^2_{L^2}dt\right)\exp\left(C\int_0^T\|u\|^{\frac{2\beta}{2\beta-3}}_{L^\beta}dt\right).$$

Using the mixed norm estimate of Stokes system, we note that

$$\int_0^T\|\Delta u\|^2_{L^3}dt\le C\|u_0\|^2_{H^1}+C\int_0^T\|n(t)\|^2_{L^3}dt+C\int_0^T\|u\|^2_{L^6}\|\nabla u\|^2_{L^6}dt<\infty, \tag{4.16}$$

where we used

$$\int_0^T\|(u\cdot\nabla)u\|^2_{L^3}dt\le\int_0^T\|u\|^2_{L^6}\|\nabla u\|^2_{L^6}dt\le\|\omega\|^2_{L^\infty(0,T;L^2)}\|\nabla\omega\|^2_{L^2(0,T;L^2)}.$$

Again with aid of the estimate of Stokes system, we have

$$\int_0^T\|\Delta u\|^2_{L^4}dt\le C\|u_0\|^2_{H^2}+C\int_0^T\|n(t)\|^2_{L^4}dt+C\int_0^T\|u\|^2_{L^6}\|\nabla u\|^2_{L^{12}}dt<\infty, \tag{4.17}$$

where we used

$$\int_0^T\|(u\cdot\nabla)u\|^2_{L^4}dt\le\int_0^T\|u\|^2_{L^6}\|\nabla u\|^2_{L^{12}}dt\le\|\omega\|^2_{L^\infty(0,T;L^2)}\|\Delta u\|^2_{L^2(0,T;L^3)}.$$

Hence it is direct that $\|\nabla u\|_{L^2(0,T;L^\infty)} < \infty$. Next, testing n to the equation of n as in 2D case, we also have (4.5). For equation of c, we can obtain (4.6) without any modification, and therefore, it is immediate that (4.7). Using Gronwall's Lemma, we obtain

$$\sup_{t\in(0,T]}(\|n(t)\|_{L^2}^2 + \|\nabla c(t)\|_{L^2}^2) + \int_0^T \|\nabla n(t)\|_{L^2}^2 dt \le C(\|n_0\|_{L^2}^2 + \|\nabla c_0\|_{L^2}^2)$$

$$\times \exp(C(T^{1/2}\|\nabla u\|_{L^2(0,T;L^\infty)} + \|n\|_{L^2(0,T;L^\infty)}^2 + T))\int_0^T (\|\nabla u\|_{L^\infty} + C\|n\|_{L^\infty}^2 + 1)dt < \infty.$$

<u>Step 2-2</u> (Induction argument) As in 2D case, most of all estimates are the same as those given above. Therefore, we just mention different estimates compared to 2D case. Up to estimate (4.12), all estimates are exactly the same as before and however, the following is slightly different form of estimate (compare to (4.13)). Indeed, using commutator estimates,

$$\left|\sum_{|\alpha|\le m+1}\int_{\mathbb{R}^3} D^\alpha((u\cdot\nabla)c)D^\alpha c dx\right| = \left|\sum_{|\alpha|\le m+1}\int_{\mathbb{R}^3}[D^\alpha((u\cdot\nabla)c) - (u\cdot\nabla)D^\alpha c]D^\alpha c dx\right|$$

$$\le C\|\nabla u\|_{L^\infty}\|c\|_{H^{m+1}}^2 + C\|\nabla c\|_{L^3}\|u\|_{W^{m+1,6}}\|c\|_{H^{m+1}},$$

$$\le C\|\nabla u\|_{L^\infty}\|c\|_{H^{m+1}}^2 + C\|\nabla c\|_{L^2}^{\frac{1}{2}}\|\nabla c\|_{H^1}^{\frac{1}{2}}\|\nabla u\|_{H^{m+1}}\|c\|_{H^{m+1}}.$$

With the above modification, to sum up, we have

$$\frac{d}{dt}(\|n\|_{H^m}^2 + \|c\|_{H^{m+1}}^2) + \|\nabla n\|_{H^{m+1}}^2$$

$$\le C(\|\nabla c\|_{L^2}^2\|n\|_{H^m}^2 + \|n\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty} + \|n\|_{L^\infty} + \|\nabla u\|_{H^{m+1}} + 1)(\|n\|_{H^m}^2 + \|c\|_{H^{m+1}}^2) + C\|n\|_{L^\infty}^2.$$

Under the same assumption as (4.8) and (4.9), Gronwall's inequality implies that

$$\|n\|_{L^\infty(0,T;H^m)} + \|c\|_{L^\infty(0,T;H^{m+1})} + \|\nabla n\|_{L^2(0,T;H^m)} < \infty.$$

This finishes the case of 3D and therefore, proof is completed. $\square$

5 Proof of Theorem 4

In this section, we present the proof of Theorem 4. The following lemma shows weighted energy estimate and truncated energy estimate shown in [5] in case that fluid is not coupled. It is remarkable that even in the presence of fluid equations, influence of fluid does not appear. Indeed, incompressibility causes cancelation of terms involving velocity of fluid, which is a crucial observation for the proof of Theorem 4.

Lemma 5 *Let $\phi(\cdot)$ be an auxiliary function such that $\phi'(c) - \phi(c)\chi(c) = 0$ and K a positive number. Then, the classical solutions to* (1.11) *satisfy the following weighted energy equality* (5.1) *and truncated energy equality* (5.2)*:*

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}\right)^p\phi(c) + 4\frac{p-1}{p}\int_{\mathbb{R}^d}\phi(c)\left|\nabla\left(\frac{n}{\phi(c)}\right)^{p/2}\right|^2$$

$$= (p-1)\int_{\mathbb{R}^d} \phi^2(c)\chi(c)k(c)\left(\frac{n}{\phi(c)}\right)^{p+1}, \tag{5.1}$$

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}-K\right)_+^p \phi(c) + 2\frac{p-1}{p}\int \phi(c)\left|\nabla\left(\frac{n}{\phi(c)}-K\right)_+^{p/2}\right|^2$$

$$= (p-1)\int\left(\frac{n}{\phi(c)}-K\right)_+^{p+1}\phi^2(c)\chi(c)k(c) + (2p-1)K\int \phi^2(c)\chi(c)k(c)\left(\frac{n}{\phi(c)}-K\right)_+^p$$

$$+ pK^2\int \phi(c)^2\chi(c)k(c)\left(\frac{n}{\phi(c)}-K\right)_+^{p-1}. \tag{5.2}$$

Proof. We note first that

$$\frac{d}{dt}\frac{n}{\phi} = \frac{n_t\phi - n\phi' c_t}{\phi^2} = \frac{1}{\phi}(\Delta n - \nabla\cdot(n\chi\nabla c) - u\cdot\nabla n) - \frac{n\phi' c_t}{\phi^2}.$$

Due to $\nabla\cdot u = 0$ and $\phi' = \phi\chi$, we also observe that

$$\Delta n - \nabla\cdot(n\chi\nabla c) - u\cdot\nabla n = \nabla\cdot\left(\phi\nabla\left(\frac{n}{\phi}\right) - \frac{n}{\phi}\phi u\right)$$

and therefore, it follows that

$$\frac{d}{dt}\frac{n}{\phi} = \frac{1}{\phi}\left[\nabla\cdot\left(\phi\nabla\left(\frac{n}{\phi}\right) - \frac{n}{\phi}\phi u\right) - n\chi c_t\right]. \tag{5.3}$$

Testing $p(\frac{n}{\phi})^{p-1}\phi$ to (5.3) and using the integration by parts, we obtain

$$\frac{d}{dt}\int\left(\frac{n}{\phi}\right)^p\phi = \int p\left(\frac{n}{\phi}\right)^{p-1}\left[\nabla\cdot\left(\phi\nabla\left(\frac{n}{\phi}\right) - \frac{n}{\phi}\phi u\right) - n\chi c_t\right] + \left(\frac{n}{\phi}\right)^p\phi\chi c_t dx$$

$$= \int p\left(\frac{n}{\phi}\right)^{p-1}\nabla\cdot\left(\phi\nabla\left(\frac{n}{\phi}\right)\right)dx - \int p\left(\frac{n}{\phi}\right)^{p-1}\nabla\cdot\left(\frac{n}{\phi}\phi u\right)dx - (p-1)\int\left(\frac{n}{\phi}\right)^p\phi\chi c_t dx. \tag{5.4}$$

We estimate separately each term in (5.4).

$$\int p\left(\frac{n}{\phi}\right)^{p-1}\nabla\cdot\left(\phi\nabla\left(\frac{n}{\phi}\right)\right) = -p(p-1)\int\phi\left(\frac{n}{\phi}\right)^{p-2}\left|\nabla\left(\frac{n}{\phi}\right)\right|^2 dx, \tag{5.5}$$

$$\begin{aligned} p\int\left(\frac{n}{\phi}\right)^{p-1}\nabla\cdot\left(\frac{n}{\phi}\phi u\right)dx &= p\int\left(\frac{n}{\phi}\right)^{p-1}\left(\frac{n}{\phi}\right)\nabla\cdot(\phi u) + \left(\frac{n}{\phi}\right)^{p-1}\nabla\left(\frac{n}{\phi}\right)\cdot\phi u dx \\ &= p\int\left(\frac{n}{\phi}\right)^p\nabla\cdot(\phi u)dx + \int\nabla\left(\frac{n}{\phi}\right)^p\cdot\phi u dx = (p-1)\int\left(\frac{n}{\phi}\right)^p\nabla\cdot(\phi u)dx, \end{aligned} \tag{5.6}$$

$$\begin{aligned} (p-1)\int\left(\frac{n}{\phi}\right)^p\phi\chi c_t dx &= -(p-1)\int\left(\frac{n}{\phi}\right)^p\phi(\chi u\cdot\nabla c + kn)dx \\ &= -(p-1)\int\left(\frac{n}{\phi}\right)^p\phi' u\cdot\nabla c dx - (p-1)\int\left(\frac{n}{\phi}\right)^p\phi\chi kn dx \\ &= -(p-1)\int\left(\frac{n}{\phi}\right)^p\nabla\cdot(u\phi)dx - (p-1)\int\phi^2\left(\frac{n}{\phi}\right)^{p+1}\chi k dx. \end{aligned} \tag{5.7}$$

Adding up (5.5)-(5.7), we obtain (5.1). By the Sobolev inequality, for any p with $\max\{1, d/2-1\} \le p < \infty$, it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^d} (\frac{n}{\phi(c)})^p \phi(c) \le (p-1) \left\| \nabla (\frac{n}{\phi(c)})^{\frac{p}{2}} \right\|^2_{L^2(\mathbb{R}^d)} \cdot \left[\tilde{C}(d) K_1 \left\| \phi^{2/d}(c) (\frac{n}{\phi(c)}) \right\|_{L^{d/2}(\mathbb{R}^d)} - \frac{4}{p} \right], \tag{5.8}$$

as long as $0 \le c \le \|c_0\|_{L^\infty}$ and $\sup_{0\le c \le \|c_0\|_{L^\infty}} \phi^2(c)\chi(c)k(c) := K_1 < \infty$. For the truncated inequality, we proceed similar computations as in those of the weighted equality. We obtain

$$\frac{d}{dt}\int \left(\frac{n}{\phi} - K\right)_+^p \phi = \int p \left(\frac{n}{\phi} - K\right)_+^{p-1} \frac{d}{dt}\left(\frac{n}{\phi}\right)\phi + \left(\frac{n}{\phi} - K\right)_+^p \phi' c_t$$

$$= \int p \left(\frac{n}{\phi} - K\right)_+^{p-1} \nabla\cdot \left(\phi \nabla \left(\frac{n}{\phi}\right) - \frac{n}{\phi}\phi u\right) - p\int \left(\frac{n}{\phi} - K\right)_+^{p-1} n\chi c_t + \int \left(\frac{n}{\phi} - K\right)_+^p \phi\chi c_t.$$

We note that integration by parts yields

$$\int p \left(\frac{n}{\phi} - K\right)_+^{p-1} \nabla\cdot \left(\phi\nabla\left(\frac{n}{\phi}\right)\right) dx = -p(p-1)\int \phi \left(\frac{n}{\phi} - K\right)_+^{p-2} \left|\nabla\left(\frac{n}{\phi}\right)\right|^2 dx. \tag{5.9}$$

With the aid of replacement of $\frac{n}{\phi}$ by $\left(\frac{n}{\phi} - K\right) + K$ and integration by parts, it is direct that

$$-p\int \left(\frac{n}{\phi} - K\right)_+^{p-1} \nabla\cdot\left(\frac{n}{\phi}\phi u\right) = -pK\int \left(\frac{n}{\phi} - K\right)_+^{p-1} \nabla\cdot(\phi u)$$

$$- (p-1)\int \left(\frac{n}{\phi} - K\right)_+^{p} \nabla\cdot(\phi u). \tag{5.10}$$

It is also straightforward that

$$p\int \left(\frac{n}{\phi} - K\right)_+^{p-1} n\chi\nabla\cdot(uc) - \int \left(\frac{n}{\phi} - K\right)_+^{p} \phi\chi\nabla\cdot(uc)dx$$

$$= p\int \left(\frac{n}{\phi} - K\right)_+^{p-1} \left(\frac{n}{\phi} - K + K\right)\phi\chi\nabla\cdot(uc) - \int \left(\frac{n}{\phi} - K\right)_+^{p} \phi\chi\nabla\cdot(uc)dx$$

$$= (p-1)\int \left(\frac{n}{\phi} - K\right)_+^{p} \phi\chi\nabla\cdot(uc)dx + pK\int \left(\frac{n}{\phi} - K\right)_+^{p-1} \phi\chi\nabla\cdot(uc)dx \tag{5.11}$$

As noticed earlier, due to $\phi\chi\nabla\cdot(uc) = u\phi'\nabla c = \nabla\cdot(\phi u)$, (5.10) and (5.11) are cancelled out each other. We also observe that

$$p\int \left(\frac{n}{\phi} - K\right)_+^{p-1} n^2\chi k = p\int \left(\frac{n}{\phi} - K\right)_+^{p-1} \left(\frac{n}{\phi} - K + K\right)^2 \chi k\phi^2$$

$$= p\int \left(\frac{n}{\phi} - K\right)_+^{p+1} \phi^2\chi k + 2pK\int \left(\frac{n}{\phi} - K\right)_+^{p} \phi^2\chi k + pK^2\int \left(\frac{n}{\phi} - K\right)_+^{p-1} \phi^2\chi k, \tag{5.12}$$

$$-\int \left(\frac{n}{\phi} - K\right)_+^{p} \phi\chi k n = -\int \left(\frac{n}{\phi} - K\right)_+^{p+1} \phi^2\chi k - K\int \left(\frac{n}{\phi} - K\right)_+^{p} \phi^2\chi k. \tag{5.13}$$

Summing up (5.9)-(5.13), we obtain (5.2). This completes the proof. □

With the help of Lemma 5, the remaining procedures of the proof of Theorem 4 are almost identical with those in [20] and, however, we give the sketch of the proof for clarity.

Sketch of proof of Theorem 4. Let $\nu(t)$, $\eta(t)$ be auxiliary functions, which will be specified later and $\phi(c)$ given in Lemma 5. We define a truncated energy $E(\xi)$ by

$$E(\xi) := \sup_{0\le t\le T}\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+^p + 2\frac{p-1}{p}\int_0^T\int\left|\nabla\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+^{p/2}\right|^2.$$

Using (5.2) and following similar procedures as in the proof of Theorem 2, it follows that under the condition $\eta(0) > \|n_0\|_{L^\infty}$

$$\begin{aligned}\phi_{min}E(\xi) \le &\sup_{0\le t\le T}\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+^p\phi(c)\\ &+2\frac{p-1}{p}\int_0^T\int_{\mathbb{R}^d}\phi(c)\left|\nabla\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+^{p/2}\right|^2\\ \le &-\xi\int_0^T\dot\eta(t)\int_{\mathbb{R}^d}\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+^{p-1}\phi(c)\\ &+(2p-1)\xi\int_0^T\eta(t)\int_{\mathbb{R}^d}\phi^2(c)\chi(c)\kappa(c)\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+^p\\ &+p\xi^2\int_0^T\eta(t)^2\int_{\mathbb{R}^d}\phi^2(c)\kappa(c)\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+^{p-1}.\end{aligned} \tag{5.14}$$

By the sobolev embedding it holds that

$$\left\|\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+\right\|^p_{L^q([0,T]\times\mathbb{R}^d)} \le CE(\xi), \quad q=p(d+2)/d. \tag{5.15}$$

On the other hands, we define the level set energy

$$U(\xi) := \int_0^T\int\nu(t)\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+^p dxdt.$$

Differentiating in ξ,

$$U'(\xi) = -\int_0^T\int p\nu(t)\eta(t)\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+^{p-1}dxdt.$$

Interpolating $p-1<p<q$, we have

$$U(\xi)\le\left(\int_0^T\int\nu(t)^{\frac{p-1}{p(1-\theta)}}\left(\frac{n}{\phi(c)}-\xi\eta(t)\right)_+^{p-1}dxdt\right)^\alpha E(\xi)^\theta,$$

for $\theta=\frac{d+2}{d+2p}$, $\alpha=\frac{2p}{d+2p}$. Let the auxiliary functions $\nu(\xi),\eta(\xi)$ satisfy that

$$\nu(t)^{\frac{p-1}{p(1-\theta)}}+|\dot\eta(t)|\le C_1\nu(t)\eta(t), \quad \eta(c)\le C_2\nu(t).$$

Then $E(\xi)$, $U(\xi)$ satisfy the following differential inequalities:

$$\begin{cases} U(\xi) \le C_1^{\alpha} |U'(\xi)|^{\alpha} E(\xi)^{\theta}, \\ E(\xi) \le \xi |U'(\xi)|(C_1 + C_2\xi) + C_2 \xi U(\xi), \end{cases} \tag{5.16}$$

Firstly we choose $\nu(t) = (1+t)^{1-\frac{d}{2p}}$ and $\eta(t) = (1+t)^{-1}$ with $C_1 = (1+T)^{1-\frac{d}{2p}}$. Working (5.16) with $G(\xi) = U(\xi)^a$ for some $0 < a < 1$, we arrive at $G(\xi)$ vanishing for a finite ξ_1(See (step 4) for Theorem 4.1 in [20]) under the condition $p > \frac{d+2}{2}$. The same holds for $U(\xi)$ and we obtain the decay

$$\|n(t)\|_{L^\infty} \le C(T)\nu(t).$$

Next, we choose $\nu(t) = \eta(t) = (1+t)^{-1}$ with $C_2 = (1+T)^{1-\frac{d}{2p}}$ to relax the initial integrability of n_0 to $p > \frac{d(d+2)}{2(d+2)}$. Up to this point, L^∞ decay of $n(t)$ depends on T for $0 < t < T$. When $\|n_0\|_{L^{\frac{d}{2}}}$ is small enough, (5.1) gives

$$\int_{\mathbb{R}^d} \left(\frac{n}{\phi(c)}\right)^{\frac{d}{2}} \phi(c) + \int_{\mathbb{R}^d} \phi(c) \left| \nabla \left(\frac{n}{\phi(c)}\right)^{d/4} \right|^2 \le C\|n_0\|_{L^{\frac{d}{2}}},$$

from which we have $\|\frac{n}{\phi(c)}\|_{L^{\frac{d+2}{2}}_{t,x}([0,1]\times\mathbb{R}^d)} < C$, and $\|n(t_0)\|_{L^{\frac{d+2}{2}}} \le C$ for $t_0 \le 1$. Now using the result for $p > \frac{d(d+2)}{2(d+2)}$ and scale invariance of the norm $\|n_0\|_{L^{\frac{d}{2}}}$ and $\|c_0\|_{L^\infty}$ under scaling (1.9), we conclude $\|n(t)\|_{L^\infty} \le \frac{C}{t}$ for a uniform constant C. This completes the proof. □

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